Propagation bounds and soft photon bounds for the massless spin-boson model

W. De Roeck[1]

Institut für Theoretische Physik
Universität Heidelberg
Philosophenweg 16,
D69120 Heidelberg, Germany

A. Kupiainen[2]

Department of Mathematics
University of Helsinki
P.O. Box 68, FIN-00014, Finland

Abstract

We consider generalized versions of the massless spin-boson model. We prove detailed bounds on the number of bosons in certain spatial regions (propagation bounds) and on the number of bosons with low momentum (soft photon bounds). This work is an extension of our earlier work in [1]. Together with the results in [1], the bounds of the present paper suffice to prove asymptotic completeness, as we describe in [4].

1 Model and result

This paper provides technical tools to prove asymptotic completeness for some models of quantum field theory with massless bosons. These tools complement those developed in [1] and also their proof is to a large extent parallel to the latter. Therefore, we refer the reader to [1] for an extended motivation of the model and to [4] for a discussion of asymptotic completeness. We first introduce the model and state the result, and then, in Section 1.4, we briefly discuss the results in this paper.

1.1 The model

Our model consists of a small system (atom, spin) coupled to a free bosonic field. The Hilbert space of the total system is

$$\mathscr{H} = \mathscr{H}_{\mathrm{S}} \otimes \mathscr{H}_{\mathrm{F}} \tag{1.1}$$

where $\mathscr{H}_{\mathrm{S}}$, the atom/spin space (S for 'small system'), is finite dimensional, $\mathscr{H}_{\mathrm{S}} \sim \mathbb{C}^n$ for some $n < \infty$. The field space $\mathscr{H}_{\mathrm{F}}$ is the bosonic Fock space $\Gamma(\mathfrak{h})$ built from the single particle space $\mathfrak{h} = L^2(\mathbb{R}^d)$:

$$\mathscr{H}_{\mathrm{F}} = \Gamma(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} P_{\mathrm{Symm}} \mathfrak{h}^{\otimes n} \tag{1.2}$$

[1] email: `w.deroeck@thphys.uni-heidelberg.de`, supported by the DFG
[2] email: `antti.kupiainen@helsinki.fi`, supported by ERC and Academy of Finland

with P_{Symm} is the projection to symmetric tensors and $\mathfrak{h}^{\otimes_0} \equiv \mathbb{C}$. The total Hamiltonian is of the form

$$H = H_{\mathrm{S}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{F}} + H_{\mathrm{I}} \tag{1.3}$$

where

i.) H_{S} is a hermitian matrix acting on $\mathscr{H}_{\mathrm{S}}$.

ii.) H_{F} is the Hamiltonian of the free field, given by

$$H_{\mathrm{F}} = \int_{\mathbb{R}^d} \mathrm{d}k |k| a_k^* a_k \tag{1.4}$$

where a_k^*, a_k are the creation/annihilation operators of a mode $k \in \mathbb{R}^d$ satisfying the 'Canonical Commutation Relations'

$$[a_k, a_{k'}^*] = \delta(k - k').$$

We omit a precise definition of these field operators (in fact, operator-valued distributions) since this belongs to standard lore in mathematical physics.

iii.) The coupling H_{I} is of the form

$$H_{\mathrm{I}} = \lambda D \otimes \Phi(\phi) \tag{1.5}$$

where $D = D^*$ is a matrix acting on $\mathscr{H}_{\mathrm{S}}$, $\lambda \in \mathbb{R}$ is a coupling constant, $\phi \in L^2(\mathbb{R}^d)$ is a "form factor" that imposes some infrared and ultraviolet regularity in the model and Φ is the self-adjoint field operator

$$\Phi(\phi) = \int_{\mathbb{R}^d} \mathrm{d}k \, (\hat{\phi}(k) a_k^* + \overline{\hat{\phi}(k)} a_k) \tag{1.6}$$

and $\hat{\phi}$ denotes the Fourier transform of ϕ.

If the form factor ϕ satisfies

$$\int \mathrm{d}k |\hat{\phi}(k)|^2 (1 + \frac{1}{|k|}) < \infty \tag{1.7}$$

then that the term H_{I} is relatively bounded w.r.t. H_{F} with arbitrarily small relative bound and therefore the Hamiltonian H in (1.3) is self-adjoint on the domain of H_{F} by Kato-Rellich. Hence the unitary dynamics $\mathrm{e}^{-\mathrm{i}tH}$ is well-defined and we set $\Psi_t = \mathrm{e}^{-\mathrm{i}tH} \Psi_0$ with $\Psi_t, \Psi_0 \in \mathscr{H}$. A lot of work has been devoted to this model, in particular to its spectral theory, but we do not discuss this here. Instead, references are collected in [1, 4].

1.2 Assumptions

We describe now our assumptions on the form factor. Its infrared (small Fourier mode k) behavior determines temporal correlations in the model and some regularity near $k = 0$ is needed. Roughly we need to assume

$$\hat{\phi}(k) \sim |k|^{-\frac{d-2-\alpha}{2}}$$

with some $\alpha > 0$ as $|k| \to 0$.

Definition 1.1. Let $0 < \alpha < 1$. We define the subspace $\mathfrak{h}_\alpha \subset \mathfrak{h}$ to consist of $\psi \in \mathfrak{h}$ such that $\hat{\psi} \in C_0^3(\mathbb{R}^d \setminus \{0\})$ and, for all multi indices m with $|m| \leq 3$.

$$|\partial_k^m \hat{\psi}(k))| \leq C|k|^{(\beta-d+2)/2-|m|}. \tag{1.8}$$

for some $\beta > \alpha$ and $C < \infty$.

In the following two assumptions, we fix once and for all the form factor, the dimension d, and the operators H_S and D. These choices and assumptions are assumed to hold throughout the article and they will not be repeated.

The first assumption controls the infrared behaviour of the model

Assumption 1 (α-Infrared regularity). *The form factor ϕ is in $\mathfrak{h}_\alpha$ and the dimension $d \geq 3$.*

The second assumption ensures that the coupling is effective. The necessity of an assumption like this can be understood immediately. If $\hat{\phi} = 0$ identically, then the atom and field are not coupled and no dissipative effects can be expected. While that would in fact not literally invalidate the results of the present paper, it certainly would destroy the results in [1] whose proof is largely parallel to those in this paper.

Assumption 2 (Fermi Golden Rule). *We assume that the spectrum of H_S is non-degenerate (all eigenvalues are simple) and we let $e_0 := \min \sigma(H_\mathrm{S})$ (atomic ground state energy). Most importantly, we assume that for any eigenvalue $e \in \sigma(H_\mathrm{S}), e \neq e_0$, there is a sequence $e(i), i = 1, \ldots, n$ of eigenvalues such that*

$$e = e(1) > e(2) > \ldots > e(n) = e_0, \qquad \textit{and} \qquad \forall i = 1, \ldots, n-1 : j(e(i), e(i+1)) > 0 \tag{1.9}$$

with $j(\cdot, \cdot)$ given by

$$j(e, e') := 2\pi \operatorname{Tr}[P_e D P_{e'} D P_e] \int_{\mathbb{R}^d} \mathrm{d}k \, \delta(|k| - (e - e'))|\hat{\phi}(k)|^2, \tag{1.10}$$

where the RHS is well-defined since $\hat{\phi}$ is continuous away from 0.

1.3 Results

We now state our main results.

To choose appropriate initial states, we introduce the *Weyl operator* $\mathcal{W}(\psi), \psi \in \mathfrak{h}$

$$\mathcal{W}(\psi) = \mathrm{e}^{\mathrm{i}\Phi(\psi)}, \tag{1.11}$$

with the (Segal) field operator $\Phi(\psi)$ as in (1.6), and define the dense subspace

$$\mathcal{D}_\alpha := \operatorname{Span}\{\psi_\mathrm{S} \otimes \mathcal{W}(\psi)\Omega, \quad \psi \in \mathfrak{h}_\alpha, \psi_\mathrm{S} \in \mathscr{H}_\mathrm{S}\} \tag{1.12}$$

The density of $\mathcal{D}_\alpha$ in $\mathcal{H}$ follows from the density of $\mathfrak{h}_\alpha$ in $\mathfrak{h}$. We will choose the initial vector $\Psi_0 \in \mathcal{D}_\alpha$ and we write $\Psi_t = \mathrm{e}^{-itH}\Psi_0$.

Fix a C^∞ function $\theta : \mathbb{R}^d \to [0, 1]$, having compact support in the ball centered at origin with radius $m_\theta < 1$. Since θ will be used to localize both in real $x-$space and in Fourier

$k-$space we use the notation $\theta(x)$ for the multiplication operator and $\theta(k)$ for the Fourier multiplier. To any self-adjoint operator b on $\mathfrak{h}$, we associate its second quantization $\mathrm{d}\Gamma(b)$, a self-adjoint operator on $\Gamma(\mathfrak{h})$, and we also write $\mathrm{d}\Gamma(b)$ for $\mathbb{1}\otimes\mathrm{d}\Gamma(b)$, acting on $\mathscr{H}$.

In the statement of our theorems, C denotes constants that depend on Ψ_0, θ, α, the dimension d, and the parameters of the Hamiltonian (1.3), i.e. the form factor ϕ and the operators H_{S}, D, but not on λ.

Theorem 1.1 (Soft photon bound). *There exists $\lambda_0 > 0$ such that for all λ with $|\lambda| \leq \lambda_0$,*

$$\sup_{t\geq 0} |\langle \Psi_t, \mathrm{d}\Gamma(\theta(k/\delta)\Psi_t\rangle| \leq C\delta^{\alpha/2} \tag{1.13}$$

for any $\delta > 0$, smooth θ as above, and $\Psi_0 \in \mathcal{D}_\alpha$.

Remark 1.2. *This result complements the photon bound in [1], where we proved that*

$$\sup_{t\geq 0} |\langle \Psi_t, N\Psi_t\rangle| \leq C \tag{1.14}$$

with $N = \mathrm{d}\Gamma(\mathbb{1})$ the number operator. Although the infrared condition in that paper is slightly different, an obvious application of Lemma A.1 in Appendix A allows to derive that condition from our present infrared condition, i.e. from Assumption 1.

Remark 1.3. *Inspecting the proof, we see that the bound $\delta^{\alpha/2}$ can be replaced by $\delta^{\alpha'}$, for any $\alpha' < \alpha$, at the cost of making the constant C dependent on α'.*

Theorem 1.4 (Propagation bound). *Let $|\lambda| \leq \lambda_0$ as in Theorem 1.1 and fix an initial state $\Psi_0 \in \mathcal{D}_\alpha$ and a smooth θ, as in Theorem 1.1. For any 'cutoff time' $t_c \geq |\lambda|^{-2}$, the limit*

$$a(t_c, \theta) := \lim_{t\to\infty} \langle \Psi_t, \mathrm{d}\Gamma(\theta(x/t_c))\Psi_t\rangle \tag{1.15}$$

exists and

$$|\langle \Psi_t, \mathrm{d}\Gamma(\theta(x/t_c))\Psi_t\rangle - a(t_c, \theta)| \leq C(1+t)^{-\alpha}, \tag{1.16}$$

uniformly in t_c for $t \geq t_c \geq |\lambda|^{-2}$, and with C as in Theorem 1.1.

Remark 1.5. *The obvious interpretation of this result is that*

$$a(t_c, \theta) = \langle \Psi_{\mathrm{gs}}, \mathrm{d}\Gamma(\theta(x/t_c))\Psi_{\mathrm{gs}}\rangle \tag{1.17}$$

where Ψ_{gs} is the unique (up to scalar multiplication) ground state of H, that can indeed be proven to exist given our assumptions, see [1]. This interpretation is correct but we postpone its statement to [4] because the identification of the limit requires somehow different reasoning that does not naturally fit into the present paper.

1.4 Discussion

In [1], we established two results. On the one side, we showed that for localized observables O, i.e. those concerning the atom and the field in the neighborhood of the atom, the expectation value $\langle \Psi_t, O\Psi_t\rangle$ converges to the stationary value $\langle \Psi_{\mathrm{gs}}, O\Psi_{\mathrm{gs}}\rangle$ (assuming that a ground state Ψ_{gs} exists). On the other side, we showed that the number of emitted photons is bounded

independently in time, i.e. (1.14). The intuition dictates that those emitted photons behave as free photons once they are sufficiently far from the atom. To make this intuition precise, we find it helpful to control observables that probe the state of the field in spatial regions inside the wavefront $|x| \leq t$ (we assume here that the speed of light is 1) but far away from the atom, i.e. for example in regions of the type $c_1 t < |x| < c_2 t$ with $0 < c_1 < c_2 < 1$, as we do in Theorem 1.4. This can probably be done by existing time-indepenent operator techniques, but we prefer to modifiy slightly the time-dependent perturbation theory in [1] to obtain these results. The soft-photon bound, Theorem 1.1 could be done completely analogously to the treatment of [1], since the one-particle operator $b = \theta(k/\delta)$ is invariant under the free photon dynamics, but to make the present paper more streamlined, we treat it analogously to Theorem 1.4, which concerns a non-invariant b-operator. Note that in [2], such an analogy is used to control $\mathrm{d}\Gamma(b)$ for $b = 1/|k|$, which is of course also invariant.

1.5 Notation

1.5.1 Combinatorics

We write $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $\tau, \tau' \in \mathbb{N}$, $\tau < \tau'$, we define the discrete intervals

$$I_{\tau,\tau'} := \{\tau, \tau+1, \ldots, \tau'\} \tag{1.18}$$

and $\mathfrak{B}_{\tau,\tau'}$ the set of collections of subsets of $I_{\tau,\tau'}$ i.e. $\mathfrak{B}_{\tau,\tau'}$ is the set of subsets of $2^{I_{\tau,\tau'}}$. A relevant subset of $\mathfrak{B}_{\tau,\tau'}$ is, for $j \in \mathbb{N}$,

$$\mathfrak{B}^j_{\tau,\tau'} := \{\mathcal{A} \in \mathfrak{B}_{\tau,\tau'} \,|\, \forall A \neq A' \in \mathcal{A} : \mathrm{dist}(A, A') > j\} \tag{1.19}$$

where $\mathrm{dist}(A, A') := \min_{\tau \in A, \tau' \in A'} |\tau - \tau'|$. For a collection $\mathcal{A}$, we set

$$\mathrm{Supp}\mathcal{A} := \cup_{A \in \mathcal{A}} A \tag{1.20}$$

and we need also the diameter of (finite) subsets of $\mathbb{N}$;

$$d(A) = \max A - \min A + 1, \qquad d(\mathcal{A}) := d(\mathrm{Supp}\mathcal{A}) \tag{1.21}$$

1.5.2 Hilbert and Banach spaces

For a Banach space $\mathscr{E}$, we let $\mathscr{B}(\mathscr{E})$ stand for the set of bounded operators. If $\mathscr{E}$ is a Hilbert space, we will additionally use the space of trace class operators

$$\mathscr{B}_p(\mathscr{E}) = \{O \in \mathscr{B}(\mathscr{E}), \|O\|_p < \infty\} \tag{1.22}$$

with

$$\|O\|_p = (\mathrm{Tr}\, |OO^*|^{p/2})^{1/p} \tag{1.23}$$

For the scalar product on a Hilbert space $\mathscr{E}$, we use the notation $\langle \psi, \psi' \rangle$ or $\langle \psi, \psi' \rangle_{\mathscr{E}}$. A postive operator $\rho \in \mathscr{B}_1(\mathscr{E})$ with $\mathrm{Tr}\, \rho = 1$ is called a density matrix. We also use the function

$$\langle x \rangle := \sqrt{x^2 + 1}, \tag{1.24}$$

for real numbers and self-adjoint operators.

1.5.3 Constants

We denote by c, C constants that depend only on parameters of the Hamiltonian 1.3 (but not on λ, for $|\lambda|$ small enough), the parameter α and the dimension d. The precise value of these constants can be different in different equations. Quantities that additionally depend on the initial condition Ψ_0 and the smooth function θ, are denoted by $\breve{c}, \breve{C}$.

2 Polymer Representation

In this section, we complete the first important step of our proof, namely we rewrite all quantities of interest through a polymer representation. This part of the paper is almost identical to a corresponding part in [1].

We discretize time by introducing a "mesoscopic" time scale λ^{-2}, where $\lambda > 0$ is the coupling strength. That is, we consider times of the form $t = n/\lambda^2$ with $n \in \mathbb{N}$. The discretization will be removed at the end of the argument. We study

$$Z_n(O, \rho_0) := \mathrm{Tr}\left[O\mathrm{e}^{-\mathrm{i}tH}\rho_0\mathrm{e}^{\mathrm{i}tH}\right], \qquad t = n/\lambda^2 \tag{2.1}$$

where the intial density matrix

$$\rho_0 = \rho_{\mathrm{S}} \otimes \mathcal{W}(\psi_{\ltimes})P_\Omega W^*(\psi_{\ltimes})$$

for some density matrix $\rho_{\mathrm{S}} \in \mathscr{B}_1(\mathscr{H}_{\mathrm{S}})$ and $\psi_{\ltimes} \in \mathfrak{h}_\alpha$, and the observable O is one of the following

- $O = \mathrm{d}\Gamma(b)$ with $b = \theta(x/t_c)$ or $b = \theta(k/\delta)$. We choose $t_c = \lambda^{-2}n_c$ for some $n_c \in \mathbb{N}$. Although, ultimately, we are interested in the case $t = t_c$, for the structure of the argument it pays off to distinguish between t and t_c.
- $O = \mathbb{1}$. This case is mainly included for comparison. By cyclicity of the trace, we have $Z_n(\mathbb{1}, \rho_0) = 1$.

In most intermediary steps of our analysis we will perform a partial trace over the field, thereby defining the reduced dynamics

$$Q_n\rho_{\mathrm{S}} := \mathrm{Tr}_{\mathrm{F}}\left[\mathrm{e}^{-\mathrm{i}(n/\lambda^2)L}\rho_0\right], \tag{2.2}$$

where we introduced the Liouvillian $L = \mathrm{ad}(H)$, an unbounded operator on $\mathscr{B}_1(\mathscr{H})$. Sometimes, we want to incorporate the observable into the reduced analysis as well. In that case, we write

$$Q_{b,n}\rho_{\mathrm{S}} := \mathrm{Tr}_{\mathrm{F}}\left[\mathrm{d}\Gamma(b)\mathrm{e}^{-\mathrm{i}(n/\lambda^2)L}(\rho_{\mathrm{S}} \otimes P_\Omega)\right] \tag{2.3}$$

Here the notation differs slightly from the one in [1] where the latter object was called $\breve{Q}_n$ and the notation Q_n was reserved for (2.2) with $\psi_{\ltimes} = 0$. Obviously, we have

$$Z_n(\mathbb{1}, \rho_0) = \mathrm{Tr}\, Q_n\rho_{\mathrm{S}} = 1, \qquad Z_n(\mathrm{d}\Gamma(b), \rho_0) = \mathrm{Tr}\, Q_{b,n}\rho_{\mathrm{S}} \tag{2.4}$$

The main goal of the first part of the present chapter is to find a convenient representation for Q_n and $Q_{b,n}$. The first step is to write the evolution operators as a multiplication where each factor corresponds to a 'mesoscopic' time slice of length λ^{-2}. With this in mind, we introduce

$$U_\tau : \mathscr{B}_1(\mathscr{H}) \to \mathscr{B}_1(\mathscr{H}) \tag{2.5}$$

with

$$U_\tau := \mathrm{e}^{\mathrm{i}(\tau/\lambda^2)L_{\mathrm{F}}} \mathrm{e}^{-\mathrm{i}(1/\lambda^2)L} \mathrm{e}^{-\mathrm{i}(\tau-1)/\lambda^2 L_{\mathrm{F}}}, \qquad \tau \in I_{1,n} \tag{2.6}$$

and

$$U_0\rho \; := \; \mathcal{W}(\psi_\ltimes)\rho W^*(\psi_\ltimes), \tag{2.7}$$

$$U_{n+1}\rho \; := \; \mathrm{d}\Gamma(b(n/\lambda^2))\rho, \tag{2.8}$$

where we wrote for brevity $\mathcal{W}(\psi)$ instead of $\mathbb{1} \otimes \mathcal{W}(\psi)$ and $b(t) = \mathrm{e}^{\mathrm{i}t|k|}b$. Note that U_{n+1} depends on the total macroscopic time n, which is a notational drawback of our formalism. An immediate use of these definitions is that we can write

$$Q_n\rho_{\mathrm{S}} := \mathrm{Tr}_{\mathrm{F}}\left[U_n \dots U_1 U_0(\rho_{\mathrm{S}} \otimes P_\Omega)\right] \tag{2.9}$$

$$Q_{b,n}\rho_{\mathrm{S}} := \mathrm{Tr}_{\mathrm{F}}\left[U_{n+1} U_n \dots U_1 U_0(\rho_{\mathrm{S}} \otimes P_\Omega)\right] \tag{2.10}$$

Finally, we define the the reduced dynamics

$$T : \mathscr{B}_1(\mathscr{H}_{\mathrm{S}}) \to \mathscr{B}_1(\mathscr{H}_{\mathrm{S}})$$

for (mesoscopic) time 1, starting from a product state

$$T\rho_{\mathrm{S}} := \mathrm{Tr}_{\mathrm{F}}\left[\mathrm{e}^{-\mathrm{i}(1/\lambda^2)L}(\rho_{\mathrm{S}} \otimes P_\Omega)\right]. \tag{2.11}$$

We set $T_\tau := T$ for $\tau = 1, \dots, n$ and

$$T_0 \; := \; \langle \mathcal{W}(\psi_\ltimes)\Omega, \mathcal{W}(\psi_\ltimes)\Omega\rangle \, \mathbb{1} = \mathbb{1} \tag{2.12}$$

$$T_{n+1} \; := \; \langle \Omega, \mathrm{d}\Gamma(b)\Omega\rangle \, \mathbb{1} = 0 \tag{2.13}$$

where we used $\|\mathcal{W}(\psi_\ltimes)\Omega\| = 1$. The motivation for this definition will become obvious in the next section. Finally, we set

$$B_\tau := U_\tau - T_\tau, \qquad \tau = 0, \dots, n+1. \tag{2.14}$$

The next section proposes a framework whose purpose is to write an expansion for Q_n and $Q_{b,n}$ in which the leading terms, in a precise sense, are $T_n \dots T_2 T_1 T_0 = T^n T_0$ and $T_{n+1} T^n T_0 = 0$, respectively.

2.1 Operator-valued polymers

2.1.1 Operator correlation functions

We abbreviate

$$\mathscr{R}_{\mathrm{S}} = \mathscr{B}(\mathscr{B}_1(\mathscr{H}_S)), \qquad \mathscr{R}_{\mathrm{F}} = \mathscr{B}(\mathscr{B}_1(\mathscr{H}_{\mathrm{F}})) \tag{2.15}$$

Define, for $W, W' \in \mathscr{R}_{\mathrm{S}} \otimes \mathscr{R}_{\mathrm{F}}$ the object

$$W \otimes_{\mathrm{S}} W' \in \mathscr{R}_{\mathrm{S}} \otimes \mathscr{R}_{\mathrm{S}} \otimes \mathscr{R}_{\mathrm{F}}$$

as an operator product in F-part and tensor product in S-part. Concretely, let $W = W_{\mathrm{S}} \otimes W_{\mathrm{F}}$ and $W' = W'_{\mathrm{S}} \otimes W'_{\mathrm{F}}$. Then

$$W \otimes_{\mathrm{S}} W' := W_{\mathrm{S}} \otimes W'_{\mathrm{S}} \otimes W_{\mathrm{F}} W'_{\mathrm{F}}.$$

and we extend this by linearity to arbitrary W, W'. Iterating this construction we define for $W_i \in \mathscr{R}_{\mathrm{S}} \otimes \mathscr{R}_{\mathrm{F}}$, $i = 1, \ldots, m$

$$W_m \otimes_{\mathrm{S}} \ldots\ldots \otimes_{\mathrm{S}} W_2 \otimes_{\mathrm{S}} W_1 \in (\mathscr{R}_{\mathcal{S}})^{\otimes^m} \otimes \mathscr{R}_{\mathrm{F}}.$$

Since $\mathscr{R}_{\mathrm{S}}$ is finite-dimensional these products are unambiguously defined.

We define the 'expectation'

$$\mathbb{E} : (\mathscr{R}_{\mathrm{S}})^{\otimes^m} \otimes \mathscr{R}_{\mathrm{F}} \ \to \ (\mathscr{R}_{\mathrm{S}})^{\otimes^m}$$

as

$$\mathbb{E}(W) J := \mathrm{Tr}_{\mathrm{F}}[W(J \otimes P_\Omega)], \qquad J \in (\mathscr{B}_1(\mathscr{H}_{\mathrm{S}}))^{\otimes^m}$$

Obviously, the action of $\mathbb{E}$ is extended to unbounded W satisfying $W((\mathscr{B}_1(\mathscr{H}_{\mathrm{S}}))^{\otimes^m} \otimes P_\Omega) \in \mathscr{B}_1(\mathscr{H}_{\mathrm{S}}^{\otimes^m} \otimes \mathscr{H}_{\mathrm{F}})$. An important example, with $m = 1$, is $T = \mathbb{E}(U_\tau)$.

Let $A = \{\tau_1, \tau_2, \ldots, \tau_m\} \subset \mathbb{N}$ with the convention that $\tau_i < \tau_{i+1}$ and define the 'time-ordered correlation function'

$$G_A := \mathbb{E}\left(B_{\tau_m} \otimes_{\mathrm{S}} B_{\tau_{n-1}} \otimes_{\mathrm{S}} \cdots \otimes_{\mathrm{S}} B_{\tau_1}\right) \in (\mathscr{R}_{S})^{\otimes^m} \tag{2.16}$$

Note that $G_A = 0$ when the set A is a singleton. Indeed, since $B_\tau = U_\tau - \mathbb{E}(U_\tau)$ we get $\mathbb{E}(B_\tau) = 0$. Also, $G_A = G_{A+\tau}$ because $\mathrm{e}^{-\mathrm{i}tL_{\mathrm{F}}} P_\Omega = P_\Omega$.

It will be convenient to label the $\mathscr{R}_{\mathrm{S}}$'s and to drop the subscript S (since we will rarely need $\mathscr{R}_{\mathrm{F}}$), writing simply $\mathscr{R}$ for $\mathscr{R}_{\mathrm{S}}$. Let us denote by $\mathscr{R}^{\otimes^{\mathbb{N}}}$ the linear space spanned by simple tensors $\ldots \otimes V_2 \otimes V_1$ where all but a finite number of V_j are equal to the identity $\mathbb{1}$. For finite subsets $A \subset \mathbb{N}$, we then define $\mathscr{R}_A$ as the finite-dimensional subspace of $\mathscr{R}^{\otimes^{\mathbb{N}}}$ spanned by simple tensors $\ldots \otimes V_2 \otimes V_1$ with $V_j = \mathbb{1}, j \notin A$ and we write in particular $\mathscr{R}_\tau = \mathscr{R}_{\{\tau\}}$. Let $A = \{\tau_1, \tau_2, \ldots, \tau_m\}$ with $\tau_1 < \tau_2 < \ldots < \tau_m$. Obviously, $\mathscr{R}_A$ is naturally isomorphic to $\mathscr{R}^{\otimes^m}$ by identifying the right-most tensor factor to $\mathscr{R}_{\tau_1}$, the next one to $\mathscr{R}_{\tau_2}$, etc... We denote this isomorphism from $\mathscr{R}^{\otimes^m}$ to $\mathscr{R}_A$ by $\boldsymbol{I}_A$ and we will from now on write G_A to denote $\boldsymbol{I}_A[G_A] \in \mathscr{R}_A$ since G_A acting on the 'unlabeled' space $\mathscr{R}^{\otimes^m}$ will not be used.

Consider a collection $\mathcal{A}$ of disjoint subsets of $\mathbb{N}$, then each of the spaces $\mathscr{R}_{A \in \mathcal{A}}$ is a subspace of $\mathscr{R}_{\mathrm{Supp}\mathcal{A}}$ where $\mathrm{Supp}\mathcal{A} = \cup_{A \in \mathcal{A}} A$. Given a collection of operators $K_A \in \mathscr{R}_A$, we have $\prod_A K_A \in \mathscr{R}_{\mathrm{Supp}\mathcal{A}}$. However, we prefer to denote such products by

$$\bigotimes_{A \in \mathcal{A}} K_A \in \mathscr{R}_{\mathrm{Supp}\mathcal{A}}, \tag{2.17}$$

i.e. we keep the tensor product explicit in the notation.

2.1.2 The contraction operator $\mathcal{T}$

We define the "contraction operator" $\mathcal{T} : \mathscr{R}_A \to \mathscr{R}$, by first giving its action on elementary tensors: Consider a family of operators $V_\tau \in \mathscr{R}$, and set

$$\mathcal{T}\left[\underset{\tau\in A}{\otimes} \boldsymbol{I}_\tau[V_\tau]\right] = V_{\tau_m} V_{\tau_{m-1}} \dots V_{\tau_1}, \qquad \text{where} \qquad \tau_m > \tau_{m-1} > \dots > \tau_1 \tag{2.18}$$

and then extend linearly to the whole of $\mathscr{R}_A$. On the LHS, we will from now on abbreviate $\boldsymbol{I}_\tau[V_\tau]$ by V_τ. This is a slight abuse of notation that should not cause confusion because we keep the tensor products explicit in the notation, as explained above.

By expanding $U_\tau = T \otimes 1 + B_\tau$ for every τ in the expression for the reduced dynamics (2.9), we arrive at

$$Q_n = \sum_{A\subset I_{0,n}} \mathcal{T}\left[G_A \underset{\tau\in I_{0,n}\setminus A}{\otimes} T_\tau\right]. \tag{2.19}$$

and in (2.10)

$$Q_{b,n} = \sum_{A\subset I_{0,n+1}} \mathcal{T}\left[G_A \underset{\tau\in I_{0,n+1}\setminus A}{\otimes} T_\tau\right]. \tag{2.20}$$

It is clear that, in the latter formula, only A with $n+1 \in A$ give a non-zero contribution because $T_{n+1} = 0$.

2.1.3 Connected correlations

Analogously to classical probability, we define the *connected correlation functions* or cumulants $G^c_A \in \mathscr{R}_A$ satisfying

$$G_A = \sum_{\mathcal{A}} \underset{A\in\mathcal{A}}{\otimes} G^c_A$$

where $\mathcal{A}$ run through the set of partitions of A. As in the classical setup, G^c_A can be solved from this inductively in $|A|$, i.e.

$$G^c_\tau = G_\tau, \qquad G^c_{\{\tau_1,\tau_2\}} = G_{\{\tau_1,\tau_2\}} - G^c_{\tau_2} \otimes G^c_{\tau_1} \tag{2.21}$$

$$G^c_{\{\tau_1,\tau_2,\tau_3\}} = G_{\{\tau_1,\tau_2,\tau_3\}} - \sum_{j=1,2,3} G^c_{\tau_j} \otimes G^c_{\{\tau_1,\tau_2,\tau_3\}\setminus\{\tau_j\}} - \underset{j=1,2,3}{\otimes} G^c_{\tau_j} \tag{2.22}$$

We obtain then

$$Q_n = \sum_{\mathcal{A}\in\mathfrak{B}^0_{0,n}} \mathcal{T}\left[\underset{A\in\mathcal{A}}{\otimes} G^c_A \underset{\tau\in I_{0,n}\setminus\mathrm{Supp}\mathcal{A}}{\otimes} T_\tau\right] \tag{2.23}$$

$$Q_{b,n} = \sum_{\mathcal{A}\in\mathfrak{B}^0_{0,n+1}} \mathcal{T}\left[\underset{A\in\mathcal{A}}{\otimes} G^c_A \underset{\tau\in I_{0,n+1}\setminus\mathrm{Supp}\mathcal{A}}{\otimes} T_\tau\right] \tag{2.24}$$

It is immediately clear that any contribution to the sum in (2.24) vanishes unless $n+1 \in \mathrm{Supp}\mathcal{A}$ because of $T_{n+1} = 0$.

2.1.4 Norms

Let us introduce a convenient norm on the space $\mathscr{R}_A$. For E acting on $\mathscr{B}_1(\mathscr{H}_{\mathrm{S}})$, we have the Banach space norm

$$\|E\| := \sup_{\rho\in\mathscr{B}_1(\mathscr{H}_{\mathrm{S}}),\|\rho\|_1=1} \|E(\rho)\|_1, \tag{2.25}$$

i.e. the natural operator norm on $\mathscr{B}(\mathscr{B}_1(\mathscr{H}_{\mathrm{S}}))$.

For $E \in \mathscr{R}_A$ with $1 < |A| < \infty$, we exploit that E can be written as a finite sum of elementary tensors

$$E = \sum_\nu E_\nu, \qquad E_\nu = \otimes_{\tau\in A} E_{\nu,\tau}, \qquad E_{\nu,\tau} \in \mathscr{R}_\tau, \tag{2.26}$$

to define

$$\|E\|_\diamond := \inf_{\{E_\nu\}} \sum_\nu \prod_{\tau\in A} \|E_{\nu,\tau}\| \tag{2.27}$$

where the infimum ranges over all such elementary tensor-representations of E. This norm is useful because of the following properties (trivial from the definition):

1) For any family of operators $K_{A\in\mathcal{A}}$ with $K_A \in \mathscr{R}_A$ and $\mathcal{A}$ a collection of disjoint sets, we have

$$\left\| \underset{A\in\mathcal{A}}{\otimes} K_A \right\|_\diamond \le \prod_{A\in\mathcal{A}} \|K_A\|_\diamond \tag{2.28}$$

2) For any $K_A \in \mathscr{R}_A$,

$$\|\mathcal{T}[K_A]\|_\diamond \le \|K_A\|_\diamond \tag{2.29}$$

2.2 Scalar polymers

The representations (2.23, 2.24) evoke the picture of a leading dynamics T interrupted by excitations, indexed by the sets $A \in \mathcal{A}$, and with operator valued weights G^{c}_A. We will now construct a similar representation, but with scalar weights. We exploit the dissipativity of the model, captured in the upcoming lemma. For operators $W' \in \mathscr{B}_1(\mathscr{H}_{\mathrm{S}}), W \in \mathscr{B}(\mathscr{H}_{\mathrm{S}})$, we write $|W'\rangle\langle W|$ to denote the operator in $\mathscr{R}$ acting as $S \mapsto |W'\rangle\langle W|S = W' \operatorname{Tr}(W^*S)$

Lemma 2.1. *Recall the operator $T \in \mathscr{R}$ defined in* (2.11). *It has a simple eigenvalue equal to* 1, *corresponding to the spectral projector $R = |\eta\rangle\langle \mathbb{1}|$, with η a density matrix, such that*

$$\|T^m - R\| \le C\mathrm{e}^{-gm} \tag{2.30}$$

for some $g > 0$.

This is Lemma 2.3 1) in [1] specialized to the case $\kappa = 0$. We exploit this to split

$$T = R + T^\perp, \tag{2.31}$$

where $T^\perp := T - R$ and we have

$$RT^\perp = T^\perp R = 0. \tag{2.32}$$

Analogously, we define $T_0 = TR + T_0^\perp = R + (\mathbb{1} - R)$ (since $T_0 = \mathbb{1}$) so that (2.32) also holds for T_0. We will insert these decompositions into the expansions (2.23, 2.24). The following definition provides the tools for this

Definition 2.1 (Fusions). Let $\mathcal{A} \in \mathcal{B}^0_{0,n}$ and let $\mathcal{J} \in \mathfrak{B}^1_{0,n}$ such that all $J \in \mathcal{J}$ are intervals. We say that a pair $(\mathcal{A}, \mathcal{J})$ is a *fusion* if

i.) $\mathrm{Supp}\mathcal{A} \cap \mathrm{Supp}\mathcal{J} = \emptyset$.

ii.) $\mathrm{dist}(I_{0,n} \setminus \mathrm{Supp}(\mathcal{A} \cup \mathcal{J}), \mathrm{Supp}\mathcal{J}) > 1$.

iii.) The following undirected graph $\Gamma(\mathcal{A}, \mathcal{J})$ is connected. Its vertex set is the disjoint union $\mathcal{A} \sqcup \mathcal{J}$, and its edges are (A, J) with $A \in \mathcal{A}, J \in \mathcal{J}, \mathrm{dist}(A, J) = 1$ and (A, A') with $A, A' \in \mathcal{A}, \mathrm{dist}(A, A') = 1$.

The set of fusions is denoted by $\mathfrak{S}^f_n$.

Remark 2.2. *The only fusions $(\mathcal{A}, \mathcal{J})$ with $\mathcal{A} = \emptyset$ are $(\emptyset, \emptyset)$ and $(\emptyset, \{I_{0,n}\})$*

Define now, for a fusion $(\mathcal{A}, \mathcal{J})$,

$$V((\mathcal{A}, \mathcal{J})) := \underset{A \in \mathcal{A}}{\otimes} G^c_A \underset{\tau \in \mathrm{Supp}\mathcal{J}}{\otimes} T^{\perp}_{\tau}, \tag{2.33}$$

as an operator in $\mathscr{R}_{\mathrm{Supp}(\mathcal{A} \cup \mathcal{J})}$, and

$$\Sigma V(A) := \sum_{(\mathcal{A}, \mathcal{J}) \in \mathfrak{S}^f_n : \mathrm{Supp}(\mathcal{A} \cup \mathcal{J}) = A} V((\mathcal{A}, \mathcal{J})) \tag{2.34}$$

We can now regroup terms in (2.23) corresponding to the fusions with the same support: This leads to

$$Q_n = \sum_{\mathcal{A} \in \mathfrak{B}^1_{0,n}} \mathcal{T}\left[\underset{\tau \in (\mathrm{Supp}\mathcal{A})^c}{\otimes} R_\tau \underset{A \in \mathcal{A}}{\otimes} \Sigma V(A) \right]. \tag{2.35}$$

where $(\mathrm{Supp}\mathcal{A})^c = I_{0,n} \setminus \mathrm{Supp}\mathcal{A}$. We refer the reader to [1] for a step by step derivation of this formula.

Note that since $\mathcal{A} \in \mathfrak{B}^1_{0,n}$ the sets $A \in \mathcal{A}$ above are non-adjacent i.e. their mutual distances are at least 2. Hence, for any $\mathcal{A}$ in the formula above, all τ that are adjacent to the set $\mathrm{Supp}\mathcal{A}$ carry the rank-one operator R. A pictorial way to phrase this is that any $\mathcal{A}$ in (2.35) is surrounded by projections R, except possibly at the boundaries of the interval $I_{0,n}$. We exploit this by defining

$$\hat{v}(A) := \mathcal{T}\left[\Sigma V(A) \bigotimes_{\tau \in I_{0,n} \setminus A} R_\tau \right], \qquad \hat{v}(A) \in \mathscr{R} \tag{2.36}$$

As argued above, we note that $\hat{v}(A)$ is a multiple of R unless $0 \in A$ and/or $n \in A$. Finally, we recall that $R = |\eta\rangle\langle \mathbb{1}|$ and define

$$v(A) := \begin{cases} \langle \mathbb{1}, \hat{v}(A)\eta \rangle & 0 \notin A \\ \langle \mathbb{1}, \hat{v}(A)\rho_{\mathrm{S},0} \rangle & 0 \in A \end{cases} \tag{2.37}$$

With these definitions, one can check that we obtain

$$Z_n(\mathbb{1}, \rho_0) = \mathrm{Tr}\, Q_n \rho_{\mathrm{S}} = \sum_{\mathcal{A} \in \mathfrak{B}^1_{0,n}} \prod_{A \in \mathcal{A}} v(A) \tag{2.38}$$

where we have used the fact that $\mathrm{Tr}\,\rho_{\mathrm{S}} = \langle \mathbb{1}, \rho_{\mathrm{S}} \rangle = 1$ to simplify the formula. Again, a more detailed derivation can be found in [1] (compared to the corresponding expression in [1] the factors $k_{\ltimes} k_{\rtimes}$ are missing, $k_{\ltimes}$ is missing because $\mathrm{Tr}\,\rho_{\mathrm{S}} = 1$ and $k_{\rtimes}$ is missing because we don't have an observable consisting of Weyl-operators). In the special case where $\rho_0 = \eta \otimes P_\Omega$, (2.38) reduces to

$$Z_n(\mathbb{1}, \eta \otimes P_\Omega) = \sum_{\mathcal{A} \in \mathfrak{B}^1_{1,n}} \prod_{A \in \mathcal{A}} v(A) \tag{2.39}$$

because in that case $v(A) = 0$ whenever $0 \in A$. This follows from $\psi_{\ltimes} = 0$ and $T^{\perp}\eta = 0$.

Remark 2.3. *Finally, note that fusions $(\mathcal{A}, \mathcal{J})$ with $n \in \mathrm{Supp}\mathcal{J}$ do not contribute to $v(\cdot)$. Indeed, they contribute to $\hat{v}(\cdot)$ an operator of the form $T^{\perp}K$ for some $K \in \mathscr{R}$, but we have*

$$\mathrm{Tr}(T^{\perp} K \rho) = \mathrm{Tr}\, T K \rho - \mathrm{Tr}\, R K \rho = 0$$

because T and R conserve the trace. In particular, by Remark 2.2, fusions with $\mathcal{A} = \emptyset$ do not contribute.

It remains to generalize this formula to the case where we have the observable $\mathrm{d}\Gamma(b)$. As already indicated, this is taken care of by defining the boundary element $n+1$. One could generalize the concepts above, like fusions, to include this element in an appropriate way, but we prefer not to do this, the reason being that the boundary element $n+1$ behaves in a very distinct way. Instead, we proceed as follows: Fix a fusion $(\mathcal{A}, \mathcal{J})$ and a set $A \in \mathcal{A}$. We modifiy the collection $\mathcal{A}$ by replacing the set A by $A \cup \{n+1\}$ and calling the obtained collection $\mathcal{A}_A$, i.e.

$$\mathcal{A}_A := (\mathcal{A} \setminus \{A\}) \cup \{A \cup \{n+1\}\} \tag{2.40}$$

Note that then the operator $V((\mathcal{A}_A, \mathcal{J}))$, defined by (2.33), remains meaningful as an operator on $\mathscr{R}_{\mathrm{Supp}(\mathcal{A} \cup \mathcal{J}) \cup \{n+1\}}$ because G^c_A with $n+1 \in A$ is well-defined. Then we set

$$\Sigma V(A' \cup \{n+1\}) := \sum_{(\mathcal{A}, \mathcal{J}) \in \mathfrak{G}^f_n : \mathrm{Supp}(\mathcal{A} \cup \mathcal{J}) = A'} \sum_{A \in \mathcal{A}} V((\mathcal{A}_A, \mathcal{J})) \tag{2.41}$$

and we simply define $\hat{v}(A' \cup \{n+1\})$ and $v(A' \cup \{n+1\})$ by the relations (2.36) and (2.37). Note that we do not include the possibility that $n+1 \in \mathrm{Supp}\mathcal{J}$. This is indeed not necessary since such a contribution would necessarily vanish because $T_{n+1} = 0$, see (2.13). Now, the final expression for Z_n reads

$$Z_n(\mathrm{d}\Gamma(b), \rho_0) = \sum_{\mathcal{A} \in \mathfrak{B}^1_{0,n}} \sum_{A \in \mathcal{A}} v(A \cup \{n+1\}) \prod_{A' \in \mathcal{A}, A' \neq A} v(A') \tag{2.42}$$

2.3 Estimates on operator-valued polymers

2.3.1 Dyson expansion

We will now derive a formula for the correlation functions G^c_A in graphical terms. Recalling that $H = H_S + H_F + H_I$ we decompose $L = \mathrm{ad}(H)$ as

$$L = L_{\mathrm{F}} + L_{\mathrm{S}} + L_{\mathrm{I}} \tag{2.43}$$

and introduce

$$L_{\mathrm{I}}(s)=\begin{cases}\mathrm{e}^{\mathrm{i}sL_{\mathrm{F}}}L_{\mathrm{I}}\mathrm{e}^{-\mathrm{i}sL_{\mathrm{F}}} & s\geq 0\\ \mathrm{ad}(\Phi(\psi_{\ltimes})) & -1\leq s<0\end{cases} \tag{2.44}$$

We develop the evolution operator $\mathrm{e}^{\mathrm{i}tL}$ and the Weyl operator $\mathcal{W}(\psi_{\ltimes})$ in a standard way in a Dyson expansion, (see [1] for more details), arriving at

$$\mathrm{e}^{\mathrm{i}tL_{\mathrm{S}}}Q_n\rho_{\mathrm{S}}=\sum_{m\in\mathbb{N}}(-1)^m\int\limits_{-1\leq t_1<\ldots<t_{2m}<n/\lambda^2}\mathrm{d}t_1\ldots\mathrm{d}t_{2m}\ \mathrm{Tr}_{\mathrm{F}}\left[L_{\mathrm{I}}(t_{2m})\ldots L_{\mathrm{I}}(t_2)L_{\mathrm{I}}(t_1)(\rho_{\mathrm{S}}\otimes P_{\Omega})\right] \tag{2.45}$$

The integrand can be written in terms of the formalism developed in Section 2.1.1 with obvious modifications

$$\mathrm{Tr}_{\mathrm{F}}\left[L_{\mathrm{I}}(t_{2m})\ldots L_{\mathrm{I}}(t_2)L_{\mathrm{I}}(t_1)(\rho_{\mathrm{S}}\otimes P_{\Omega})\right]=\left(\mathcal{T}\mathbb{E}\left[L_{\mathrm{I}}(t_{2m})\otimes_{\mathrm{S}}\ldots\otimes_{\mathrm{S}}L_{\mathrm{I}}(t_2)\otimes_{\mathrm{S}}L_{\mathrm{I}}(t_1)\right]\right)\rho_{\mathrm{S}}. \tag{2.46}$$

These modifications will not be discussed here in detail (see [1]). Briefly said, we introduce copies of $\mathscr{R}$ indexed by the times $t_1,t_2,\ldots,t_m$ and labelled products of them. For example, the term $\mathbb{E}[\ldots]$ above is an element of $\mathscr{R}^{\otimes m}$ that we identify with an element of $\mathscr{R}_{\{t_1,\ldots,t_m\}}$, and the operator $\mathcal{T}$ contracts it into an element of $\mathscr{R}$. Applying Wick's theorem, one gets

$$(-1)^m\mathbb{E}\left[L_{\mathrm{I}}(t_{2m})\otimes_{\mathrm{S}}\ldots\otimes_{\mathrm{S}}L_{\mathrm{I}}(t_2)\otimes_{\mathrm{S}}L_{\mathrm{I}}(t_1)\right]=\sum_{\pi\in\mathrm{Pair}(t_1,\ldots,t_{2m})}\ \bigotimes_{(u,v)\in\pi}K_{v,u}$$

where $\mathrm{Pair}(t_1,\ldots,t_{2m})$ denotes the set of partitions of the set $\{t_1,\ldots,t_{2m}\}$ into pairs $\pi=(u,v)$, $u\leq v$ and K_w with $w=(u,v)$, $u\leq v$, is given by

$$K_w=-\mathbb{E}(L_{\mathrm{I}}(v)\otimes_{\mathrm{S}}L_{\mathrm{I}}(u)). \tag{2.47}$$

where $K_w\in\mathscr{R}^{\otimes 2}$ is identified with an element of $\mathscr{R}_{\{u,v\}}$. Substituting in (2.45) we arrive at

$$\mathrm{e}^{\mathrm{i}tL_{\mathrm{S}}}Q_n=\sum_{m\in\mathbb{N}}\int\limits_{-1\leq t_1<\ldots<t_{2m}<n/\lambda^2}\mathrm{d}t_1\ldots\mathrm{d}t_{2m}\sum_{\pi\in\mathrm{Pair}(t_1,\ldots,t_{2m})}\mathcal{T}\left[\bigotimes_{w\in\pi}K_w\right] \tag{2.48}$$

In [1] it is explained how this expression may be written as an integral in a suitable space. Consider a set whose elements are families $\underline{w}$ of pairs of times: $\underline{w}=\{w_1,w_2,\ldots,w_m\}$ with $m\geq 0$ and $w_i=(u_i,v_i)$, $u_i\leq v_i$ and $u_i,v_i\in[-1,n/\lambda^2]$. This set carries a σ-algebra and a measure $\mu(\mathrm{d}\underline{w})$ so that (2.48) becomes

$$\mathrm{e}^{\mathrm{i}tL_{\mathrm{S}}}Q_n=\int\mu(\mathrm{d}\underline{w})\mathcal{T}\left[\bigotimes_i K_{w_i}\right] \tag{2.49}$$

The same expansion can be perfomed in the presence of the observable $\mathrm{d}\Gamma(b)$.

$$\begin{aligned}\mathrm{e}^{\mathrm{i}tL_{\mathrm{S}}}Q_{b,n}&=\sum_{m\in\mathbb{N}}(-1)^m\int\limits_{-1\leq t_1<\ldots<t_{2m}<n/\lambda^2}\mathrm{d}t_1\ldots\mathrm{d}t_{2m}\ \mathrm{Tr}_{\mathrm{F}}\left[U_{n+1}L_{\mathrm{I}}(t_{2m})\ldots L_{\mathrm{I}}(t_1)(\cdot\otimes P_{\Omega})\right]\\ &=\sum_{m\in\mathbb{N}}\int\limits_{-1\leq t_1<\cdots<t_{2m}\leq n/\lambda^2}\mathrm{d}t_1\ldots\mathrm{d}t_{2m}\sum_{\pi\in\mathrm{Pair}(t_1,\ldots,t_{2m})}\sum_{(u_0,v_0)\in\pi}\mathcal{T}\left[K_{v_0,u_0|b}\bigotimes_{(u,v)\in\pi,(u,v)\neq(u_0,v_0)}K_{v,u}\right]\\ &=\int\mu(\mathrm{d}\underline{w})\sum_i\mathcal{T}\left[K_{w_i|b}\bigotimes_{j\neq i}K_{w_j}\right]\end{aligned} \tag{2.50}$$

with now additionally

$$K_{w|b} = -\mathbb{E}(U_{n+1} \otimes_{\mathrm{S}} L_{\mathrm{I}}(v) \otimes_{\mathrm{S}} L_{\mathrm{I}}(u)), \qquad w = (u, v). \tag{2.51}$$

which is an operator in $\mathscr{R}^{\otimes 3}$ that we identify with $\mathscr{R}_{\{u,v,t\}}$ where the copy $\mathscr{R}_t$ with $t = n/\lambda^2$, accomodates the operator U_{n+1}.

We proceed with the identification of $G^c(A)$ from these expansions. To do that we need to coarse grain them to the macroscopic time scale (in units of $1/\lambda^2$). Given an $s \in [-1, n/\lambda^2]$ let $[s]$ denote the smallest integer not smaller than $\lambda^2 s$ i.e. $s \in]\lambda^{-2}([s]-1), \lambda^{-2}[s]]$. Then, given $\underline{w} = \{w_1, w_2, \dots, w_m\}$ let $[\underline{w}] \subset \mathbb{N}$ be the union of the $[u_i]$ and $[v_i]$ for $w_i = (u_i, v_i)$.

The contraction operator $\mathcal{T}[\cdot]$ defined in Section 2.1.2 contracts operators so as to produce an operator in $\mathscr{R}$. We now define a contraction operator $\mathcal{T}_A$ that produces operators in $\mathscr{R}_A$. Let us consider a finite family of operators $V_{t_i} \in \mathscr{R}_{t_i}$ where the indexed times t_i satisfy $t_i < t_{i+1}$ and $[t_i] \in A$. Then we set

$$\mathcal{T}_A\left[\bigotimes_i V_{t_i}\right] := \bigotimes_{\tau \in A} \boldsymbol{I}_\tau\left[\mathcal{T}\left[\bigotimes_{j:[t_j]=\tau} V_{t_j}\right]\right] \tag{2.52}$$

and we extend by linearity to the whole of $\otimes_i \mathscr{R}_{t_i}$, obtaining $\mathcal{T}_A : \otimes_i \mathscr{R}_{t_i} \mapsto \mathscr{R}_A$. In words, $\mathcal{T}_A$ puts each operator into the right 'macroscopic' time-copy and contracts the operators within each macroscopic time-copy. Coarse graining (2.49) this way leads to the formula

$$\tilde{Y}_A G(A) Y_A = \int \mu(\mathrm{d}\underline{w}) 1_{[\underline{w}]=A} \mathcal{T}_A\left[\bigotimes_{i=1}^m K_{w_i}\right]. \tag{2.53}$$

The factors $\tilde{Y}_A$ and Y_A come from the free S-evolutions in (2.49) and the definition of $G(A)$:

$$Y_\tau = \boldsymbol{I}_\tau[\mathrm{e}^{\mathrm{i}(\tau-1)L_{\mathrm{S}}}], \quad Y_A = \bigotimes_{\tau \in A} Y_\tau, \qquad \text{and} \qquad \widetilde{Y}_\tau = \boldsymbol{I}_\tau[\mathrm{e}^{-\mathrm{i}\tau L_{\mathrm{S}}}], \quad \widetilde{Y}_A = \bigotimes_{\tau \in A} \widetilde{Y}_\tau \tag{2.54}$$

Since $\mathrm{e}^{-\mathrm{i}\tau L_{\mathrm{S}}}$ is an isometry in the operator norm of $\mathscr{B}_1(\mathscr{H}_{\mathrm{S}})$, left and right multiplication by $Y_A, \widetilde{Y}_A$ is an isometry on $\mathscr{R}_A$ in the norm $\|\cdot\|_\diamond$, and therefore $\widetilde{Y}_A$ and Y_A play no role in what follows.

The connected correlations $G^c(A)$ have similar quite obvious expressions. Given a $\underline{w}$ we can define an undirected graph $\mathcal{G}(\underline{w})$ with vertex set $[\underline{w}]$ and edges $\{\tau, \tau'\}, \tau \le \tau'$ whenever there is a pair $w_i = (u_i, v_i)$ such that $[u_i] = \tau$ and $[v_i] = \tau'$ (we include the possibility of self-edges, when $\tau = \tau'$). We have then

Lemma 2.4. *Let $A \in I_{0,n}$ and let $\mathcal{C}(A)$ be the set of $\underline{w}$ such that $[\underline{w}] = A$ and $\mathcal{G}(\underline{w})$ is connected. Then*

$$G^c(A) \cong \int \mu(\mathrm{d}\underline{w}) 1_{\mathcal{C}(A)} \mathcal{T}_A\left[\bigotimes_{i=1}^m K_{w_i}\right] \tag{2.55}$$

$$G^c(A \cup \{n+1\}) \cong \int \mu(\mathrm{d}\underline{w}) 1_{\mathcal{C}(A)} \sum_{i=1}^m \mathcal{T}_A\left[K_{w_i|b} \bigotimes_{j \neq i} K_{w_j}\right] \tag{2.56}$$

where $\cong$ detnotes an isometry in the norm $\|\cdot\|_\diamond$.

We refer to [1] for the obvious proof. Note that whenever $A \subset I_{0,n}$ is a singleton, $G^c(A)$ vanishes, but $G^c(A \cup \{n+1\})$ does not vanish. It is in this case (only) that self-edges matter, i.e. only in this case does the condition that $\mathcal{G}(\underline{w})$ be connected depend on the presence of self-edges.

By Lemma 2.4, and the properties of the norm $\|\cdot\|_\diamond$, we immediately get the bounds

$$\|G^c(A)\|_\diamond \leq \int \mu(\mathrm{d}\underline{w}) 1_{\mathcal{C}(A)} \prod_{i=1}^{m} \|K_{w_i}\|_\diamond \tag{2.57}$$

$$\|G^c(A \cup \{n+1\})\|_\diamond \leq \int \mu(\mathrm{d}\underline{w}) 1_{\mathcal{C}(A)} \sum_{i=1}^{m} \|K_{w_i|b}\|_\diamond \prod_{j \neq i} \|K_{w_j}\|_\diamond \tag{2.58}$$

2.3.2 Bounds on the operators $K_w, K_{w|b}$ and $G^c(A)$

To bound the operators K_w, we first have to address the fact that these operators are qualitatively different whenever one or both of the times $\{u, v\}$ is smaller than 0 (because then it originates from the expansion of the Weyl operator, rather than from the interaction). Let us write (recall the form factor ϕ)

$$\phi_s = 1_{s \geq 0} \mathrm{e}^{\mathrm{i}s\omega} \phi + 1_{s<0} \psi_\ltimes \tag{2.59}$$

where ω is the self-adjoint multiplication operator with the Fourier multiplier $|k|$, i.e. $\widehat{(\omega\psi)}(k) = |k|\hat{\psi}(k)$. Then we define the functions $h(u,v)$ and $h(u,v|b)$ by

$$|\lambda|^{1_{u\geq 0}+1_{v\geq 0}} h(u,v) := \|K_{v,u}\|_\diamond, \qquad |\lambda|^{1_{u\geq 0}+1_{v\geq 0}} h(u,v|b) := \|K_{v,u|b}\|_\diamond \tag{2.60}$$

where we should warn the reader that $h(u,v|b)$ depends on δ in case $b = \theta(k/\delta)$ and on t_c and the final time t in case $b = \theta(x/t_c)$. From the definition of the norm $\|\cdot\|_\diamond$ and the definition of U_{n+1} in (2.8) we have

$$h(u,v) \leq 4\|D\| |\langle \phi_v, \phi_u \rangle_{\mathfrak{h}}|, \qquad h(u,v|b) \leq 4\|D\| |\langle \phi_u, \mathrm{e}^{-\mathrm{i}\omega t} b \mathrm{e}^{\mathrm{i}\omega t} \phi_v \rangle_{\mathfrak{h}}| \tag{2.61}$$

The important properties of the functions $h(u,v), h(u,v|b)$ are collected in

Proposition 2.5 (Bounds on correlation functions). *Unless mentioned otherwise, let $u > -1$.*

1 If $u > 0$ and $s > 0$, then

$$h(u,v) = h(u+s, v+s), \qquad h(u,v|b) = h(u+s, v+s|b), \quad \textit{for } b = \theta(k/\delta) \tag{2.62}$$

If $b = \theta(x/t_c)$, then $h(u,v|b)$ depends on the final time t and in general $h(u,v|b) \neq h(u+s, v+s|b)$. We can indicate this dependence by writing $h(u,v,t|b)$, then

$$h(u,v,t|b) = h(u+s, v+s, t+s|b) \tag{2.63}$$

2

$$\int_u^\infty \mathrm{d}v \langle v-u \rangle^{1+\alpha} h(u,v) \leq C \tag{2.64}$$

3 Let $b=\theta(k/\delta)$, then

$$\int_u^\infty \mathrm{d}v \langle v-u\rangle^{1+\alpha/2} h(u,v|b) \leq \breve{C}\delta^{\alpha/2} \tag{2.65}$$

4 Let $b=\theta(x/t_c)$, then

$$\int_u^t \mathrm{d}v \langle v-u\rangle^{1+\alpha} h(u,v|b) \leq \breve{C} \tag{2.66}$$

5 Let again $b=\theta(x/t_c)$ and fix a number m_θ such that $0<m_\theta<\sup|\mathrm{Supp}\theta|<1$.

$$\int_{-1}^{t-m_\theta t_c} \mathrm{d}u \int_u^{t-t_c} \mathrm{d}v \, \langle t-m_\theta t_c-u\rangle^{\alpha} h(u,v|b) \leq \breve{C} \tag{2.67}$$

Item 1) follows immediately from the fact that $\theta(k/\delta)$ commutes with $\mathrm{e}^{\mathrm{i}u\omega}$ and the group property $\mathrm{e}^{-\mathrm{i}v\omega}\mathrm{e}^{\mathrm{i}u\omega} = \mathrm{e}^{-\mathrm{i}(v-u)\omega}$. The proofs of the other claims concern only the one-boson problem and they are of a completely different nature than the rest of this paper. Therefore, we gather those proofs in Appendix A.

2.3.3 Bounds for operator-valued polymers

The next step is to use the bounds on the h-functions and the formulae (2.57,2.58) to derive bounds on the operator-valued correlation functions G^c_A.

Consider first (2.57). Let $\mathscr{T}(A)$ be the set of $\underline{w}$ such that $[\underline{w}]=A$ and $\mathcal{G}(\underline{w})$ is a (connected) tree. Then

$$\int \mu(\mathrm{d}\underline{w}) 1_{\mathcal{C}(A)} \prod_{i=1}^m \|K_{w_i}\|_\diamond \leq \int \mu(\mathrm{d}\underline{w}') 1_{\mathscr{T}(A)} \prod_{i=1}^{m'} \|K_{w'_i}\|_\diamond \int \mu(\mathrm{d}\underline{w}'') 1_{[\underline{w}'']\subset A} \prod_{i=1}^{m''} \|K_{w''_i}\|_\diamond \tag{2.68}$$

Indeed, the pairings and integrals on the left hand side form a subset of the ones on the right hand side: since $\mathcal{G}(\underline{w})$ is connected it contains a spanning tree $\mathscr{T}$ (in general not unique) and thus there is a subset $\underline{w}'$ of $\underline{w}$ so that $\mathcal{G}(\underline{w}')=\mathscr{T}$. The remaining set of pairs $\underline{w}''$ in $\underline{w}$ meets the constraint $[\underline{w}'']\subset A$.

We first perform the intergral over $\underline{w}''$. The integrability results in Proposition 2.5 lead to the estimate

$$\int \mu(\mathrm{d}\underline{w}'') 1_{[\underline{w}'']\subset A} \prod_{i=1}^{m''} \|K_{w''_i}\|_\diamond \leq (1+C_\ltimes 1_{0\in A})\mathrm{e}^{C|A|} \tag{2.69}$$

This is explained in detail in [1]. To perform the integral over $\underline{w}'$ let us define for $\tau,\tau'\in\mathbb{N}$, $\tau<\tau'$

$$\hat{e}(\tau,\tau') = |\lambda|^{1+1_{\tau>0}} \int_{\mathrm{Dom}(\tau)} \mathrm{d}u \int_{\mathrm{Dom}(\tau')} \mathrm{d}v \, h(u,v). \tag{2.70}$$

Here we use the notation $\mathrm{Dom}(\tau) =]\lambda^{-2}(\tau-1),\lambda^{-2}\tau]$ for $\tau>0$ and $\mathrm{Dom}(0)=]-1,0]$, i.e. $\mathrm{Dom}\tau=\{s\geq -1, [s]=\tau\}$. Then

$$\int \mu(\mathrm{d}\underline{w}') 1_{\mathscr{T}(A)} \prod_{i=1}^{m'} \|K_{w'_i}\|_\diamond \leq \sum_{\mathscr{T}:\mathcal{V}(\mathscr{T})=A} \prod_{\{\tau,\tau'\}\in\mathcal{E}(\mathscr{T})} \hat{e}(\tau,\tau') \tag{2.71}$$

where the sum runs over all trees $\mathscr{T}$ whose vertex set $\mathcal{V}(\mathscr{T})$ is A, i.e. over all spanning trees on A, and $\mathcal{E}(\mathscr{T})$ is the edge set of the tree $\mathscr{T}$. It is understood that a tree does not contain self-edges. Altogether we have obtained

Lemma 2.6. *Let $A \subset I_{0,n}$, then*

$$\|G^c_A\|_\diamond \leq (1 + 1_{[0\in A]}\breve{C})\mathrm{e}^{C|A|} \sum_{\mathscr{T}:\mathcal{V}(\mathscr{T})=A} \prod_{\{\tau,\tau'\}\in\mathcal{E}(\mathscr{T})} \hat{e}(\tau,\tau') \tag{2.72}$$

In dealing with (2.58), we distinguish the cases $a)$ $[u_i] = [v_i]$ and $b)$ $[u_i] \neq [v_i]$. In case $b)$ we may assume $w_i \in \underline{w}'$ whereas in case $a)$ w_i cannot belong to a spanning tree. Defining for $\tau \leq \tau'$,

$$\hat{e}(\tau,\tau'|b) = |\lambda|^{1_{\tau>0}+1_{\tau'>0}} \int_{\mathrm{Dom}(\tau)} \mathrm{d}u \int_{\mathrm{Dom}(\tau')} \mathrm{d}v\, h(u,v|b) 1_{v\geq u} \tag{2.73}$$

we end up with

Lemma 2.7. *Let $A \subset I_{0,n}$, then*

$$\|G^c_{A\cup\{n+1\}}\|_\diamond \leq \breve{C}\mathrm{e}^{C|A|} \sum_{\mathscr{T}:\mathcal{V}(\mathscr{T})=A} \Big(\sum_{e_0=\{\tau_0,\tau_0'\}\in\mathcal{E}(\mathscr{T})} \hat{e}(\tau_0,\tau_0'|b) \prod_{\{\tau,\tau'\}\in\mathcal{E}(\mathscr{T})\setminus e_0} \hat{e}(\tau,\tau') \\ + \sum_{\tau_0\in A} \hat{e}(\tau_0,\tau_0|b) \prod_{\{\tau,\tau'\}\in\mathcal{E}(\mathscr{T})} \hat{e}(\tau,\tau') \Big) \tag{2.74}$$

To proceed, we need bounds on the $\hat{e}$ factors. They follow rather straightforwardly from the bounds on $h(u,v), h(u,v|b)$. For $\hat{e}(\tau,\tau')$, $\tau \neq 0$, we repeat the bound from [1].

$$\sum_{\tau'\in I_{1,n}\setminus\{\tau\}} \langle \tau'-\tau\rangle^{1+\alpha} \hat{e}(\tau,\tau') \leq \begin{cases} C|\lambda^2| & \tau \neq 0 \\ \breve{C}|\lambda| & \tau = 0 \end{cases} \tag{2.75}$$

To obtain this bound, we replace the sum by the integrals $\int \mathrm{d}u \int \mathrm{d}v$. For $\tau'-\tau > 1$, we gain a factor $|\lambda|^{2(1+\alpha)}$ by using $\langle\tau'-\tau\rangle^{1+\alpha} \leq |\lambda|^{2(1+\alpha)}\langle v-u\rangle^{1+\alpha}$ and item 2) of Proposition 2.5 . This factor compensates the λ^{-2} coming from he integration over u. For $\tau' = \tau+1$, this can also be done, except that there one cannot extract a factor $|\lambda|^{2\alpha}$ so that the bound (2.75) cannot be improved.

For $b = \theta(k/\delta)$, we get

$$\sum_{\tau'\in I_{0,n}} \langle\tau'-\tau\rangle^{1+\alpha/2} \hat{e}(\tau,\tau'|b) \leq \breve{C}\delta^{\alpha/2} \tag{2.76}$$

Compared to (2.75), the term $\tau = \tau'$ is now included in the sum and there is no small factor λ on the RHS. The derivation proceeds as above, but now starting from item 3) of Proposition 2.5.

For $b = \theta(x/t_c)$, we first state the analogue of (2.76), using Proposition 2.5 item 4);

$$\sum_{\tau'\in I_{0,n}} \langle\tau'-\tau\rangle^{1+\alpha} \hat{e}(\tau,\tau'|b) \leq \breve{C} \tag{2.77}$$

and also, using 2.5 item 5):

$$\sum_{\tau\leq n-m_\theta n_c} \sum_{\tau\leq\tau'\leq n} \langle n - m_\theta n_c - \tau\rangle^\alpha \hat{e}(\tau,\tau'|b) \leq \breve{C} \tag{2.78}$$

where n_c was defined at the beginning of Section 2 (recall $t_c = \lambda^{-2}n_c$).

2.4 Bounds on scalar polymers

The scalar polymer weights $v(A)$ were define in Section 2.2. We now show how to bound them.

Lemma 2.8 (Bounds on scalar polymers). 1) *For bulk polymers, i.e.* $0 \notin A$*, we have for all* $\tau \in I_{1,n}$*,*

$$\sum_{A \subset I_{1,n}: \tau \in A} \mathrm{e}^{c|A|} d(A)^{1+\alpha} |v(A)| \leq C\lambda^2, \tag{2.79}$$

and moreover $v(A) = v(A+\tau)$ *if* $A, A+\tau \in I_{1,n}$*.*

2) *For polymers containing* 0*, we have*

$$\sum_{A \subset I_{0,n}: 0 \in A} \mathrm{e}^{c|A|} d(A)^{1+\alpha} |v(A)| \leq \breve{C}|\lambda|, \tag{2.80}$$

3) *Let* $b = \theta(k/\delta)$*, then for all* $\tau \in I_{0,n}$

$$\sum_{A \subset I_{n,0}: A \ni \tau} \mathrm{e}^{c|A|} d(A)^{1+\alpha/2} |v(A \cup n+1)| \leq \breve{C}\delta^{\alpha/2}, \tag{2.81}$$

4) *Let* $b = \theta(x/t_c)$*, then for all* $\tau \in I_{0,n}$

$$\sum_{A \subset I_{0,n}: \tau \in A} \mathrm{e}^{c|A|} d(A)^{1+\alpha} |v(A \cup n+1)| \leq \breve{C}, \tag{2.82}$$

5) *Let* $b = \theta(x/t_c)$*, then*

$$\sum_{A \subset I_{0,n}: \min A \leq n - m_\theta n_c} \mathrm{e}^{c|A|} \langle n - m_\theta n_c - \min A \rangle^\alpha |v(A \cup \{n+1\})| \leq \breve{C} \tag{2.83}$$

2.4.1 Proof of Lemma 2.8

First, we treat the polymers not containing $n+1$ nor 0.

By using the definitions (2.37, 2.36, 2.34, 2.33), we can bound the polymer weight $v(A)$ by a product of $\|\cdot\|_\diamond$-norms of operators G^c_A, projections R and $T_\perp^{|J|}$, i.e.

$$|v(A)| \leq \sum_{(\mathcal{A},\mathcal{J}) \in \mathfrak{S}^f_n: \mathrm{Supp}(\mathcal{A} \cup \mathcal{J}) = A} \|R\|_\diamond^{|\mathrm{Supp}\mathcal{A}|} \prod_{A \in \mathcal{A}} \|G^c_A\|_\diamond \prod_{J \in \mathcal{J}} \|T_\perp^{|J|}\|_\diamond \tag{2.84}$$

Next, we use $\|R\| \leq C$, $\|T^m_\perp\| \leq C\mathrm{e}^{-mg}$ and the bounds (2.72) on $\|G^c_A\|_\diamond$ to get

$$|v(A)| \leq \sum_{(\mathcal{A},\mathcal{J}) \in \mathfrak{S}^f_n: \mathrm{Supp}(\mathcal{A} \cup \mathcal{J}) = A} \prod_{J \in \mathcal{J}} (C\mathrm{e}^{-|J|g}) \prod_{A \in \mathcal{A}} \sum_{\mathscr{T}: \mathcal{V}(\mathscr{T}) = A} \prod_{\{\tau,\tau'\} \in \mathcal{E}(\mathcal{T})} \hat{e}(\tau,\tau') \tag{2.85}$$

If we had allowed $0 \in A$, the bound (2.85) remains valid if we multiply the RHS by C. Indeed, the only change that we encounter is due to the factor $\|(T_\perp)_0\|$ which appears at most once.

We estimate (2.85) by viewing all sums on the RHS as a sum over certain connected graphs. Let $\mathscr{S} = \mathcal{J} \sqcup \mathcal{E}(\mathcal{T})$, i.e. we label the element of $\mathscr{S}$ as intervals (J) or edges (E). The elements of $\mathscr{S}$ are denoted by S, S' and collections of them are denoted by $\mathcal{S}$. We write $\mathrm{Supp}S$ to denote the subset of $\mathbb{N}$ defined by S, i.e. S without the interval/edge label, and $\mathrm{Supp}\mathcal{S} = \cup_S \mathrm{Supp}S$. We asign to any $S \in \mathscr{S}$ a weight $w_{\mathrm{s}}^{(\beta)}(S)$, with $\beta > 0$, as follows:

$$w_{\mathrm{s}}^{(\beta)}(S) := c(w) \times \begin{cases} \langle \tau' - \tau \rangle^{\beta} |\lambda|^{-2} \hat{e}(\tau, \tau') & S \text{ is the edge } E = \{\tau, \tau'\} \\ \mathrm{e}^{-(g/2)|J|} & S \text{ is the interval } J \end{cases} \tag{2.86}$$

where g is as in Lemma 2.1 and the constant $c(w)$ will be chosen to be sufficiently small. We define an adjacency relation $\underset{\mathrm{s}}{\sim}$ on $\mathscr{S}$ by

$$\begin{aligned} J \underset{\mathrm{s}}{\sim} E &\Leftrightarrow \mathrm{dist}(J, E) = 1 \\ E \underset{\mathrm{s}}{\sim} E' &\Leftrightarrow E \cap E' \neq \emptyset \\ J \underset{\mathrm{s}}{\sim} J' &\Leftrightarrow J = J' \end{aligned} \tag{2.87}$$

and we will also use this notation in the case where $\tau \in E$ or $0 \in J$. The constant $c(w)$ is now chosen such that the following relation holds, uniformly in λ,

$$\sum_{S \in \mathscr{S}, S \underset{\mathrm{s}}{\sim} S'} w_{\mathrm{s}}^{(\beta)}(S) \leq 1/\mathrm{e}, \qquad \text{for any } \beta \leq 1 + \alpha \tag{2.88}$$

where we used (2.75) to control the sum in the case where S is an edge E. For λ small enough, $c \leq g/2$, and C large enough, we claim

$$\mathrm{e}^{c|A|} d(A)^{1+\alpha} |v(A)| \leq \lambda^2 C \sum_{\mathcal{S} \subset \mathscr{S}} 1_{\mathrm{Supp}\mathcal{S} = A} 1_{\mathcal{S} \text{ connected}} \prod_{S \in \mathcal{S}} w_{\mathrm{s}}^{(1+\alpha)}(S) \tag{2.89}$$

where $\mathcal{S}$ connected means that the graph with vertex set $\mathcal{S}$ and edges $\{S, S'\}$ if $S \underset{\mathrm{s}}{\sim} S'$, is connected. To check (2.89), note that

i) $\langle \tau - \tau' \rangle \langle \tau' - \tau'' \rangle \leq \langle \tau - \tau'' \rangle$

ii) $v(A)$ always contains at least one factor λ^2. This is because any fusion contributing to $v(A)$ with $0 \notin A$, has $\mathcal{A} \neq \emptyset$, and any $\hat{e}(\tau, \tau')$ with $\tau', \tau > 0$ carries a factor λ^2.

iii) the notion of connectedness defined by the relation $\underset{\mathrm{s}}{\sim}$ corresponds to the one on the RHS of eq. (2.85), in the following sense: We start from a fusion $(\mathcal{A}, \mathcal{J})$ and we choose for any $A \in \mathcal{A}$, a spanning tree $\mathscr{T}_A$ on A. Then, consider the subset of $\mathscr{S}$ which consists of the edges $\cup_{A \in \mathcal{A}} \mathcal{E}(\mathscr{T}_A)$ and the intervals $J \in \mathcal{J}$. This subset is connected by the adjacency relation $\underset{\mathrm{s}}{\sim}$.

To finish the proof, we invoke a combinatorial bound stating that, provided (2.88) holds, we have

$$\sum_{\mathcal{S} \subset \mathscr{S} : \mathcal{S} \underset{\mathrm{s}}{\sim} S_0} k(\mathcal{S}) \prod_{S \in \mathcal{S}} w_{\mathrm{s}}^{(\beta)}(S) \leq 1, \qquad k(\mathcal{S}) := \sum_{\mathcal{G} \in \mathscr{G}^c(\mathcal{S})} \prod_{\{S, S'\} \in \mathcal{E}(\mathcal{G})} 1_{S \underset{\mathrm{s}}{\sim} S'} \tag{2.90}$$

where $\mathcal{S} \underset{\mathrm{s}}{\sim} S_0$ means that $S \underset{\mathrm{s}}{\sim} S_0$ for at least one $S \in \mathcal{S}$, $\mathscr{G}^c(\mathcal{S})$ is the set of connected (undirected) graphs with vertex set $\mathcal{S}$ and $\mathcal{E}(\mathcal{G})$ is the edge set of the graph $\mathcal{G}$. A more extended presentation of (a more general version of) this bound is found in Appendix A of [1], it is a standard ingredient of cluster expansions. We estimate

$$
\begin{aligned}
|\lambda|^{-2} \sum_{A:\tau\in A} \mathrm{e}^{c|A|} d(A)^{1+\alpha} |v(A)| &\leq C \sum_{\mathcal{S}\subset\mathscr{S}} 1_{\tau\in\mathrm{Supp}\mathcal{S}}\, k(\mathcal{S}) \prod_{S\in\mathcal{S}} w_{\mathrm{s}}^{(1+\alpha)}(S) \\
&\leq C \sum_{\mathcal{S}\subset\mathscr{S}} k(\mathcal{S}) \left(\sum_{\tilde\tau=-\tau-1,\tau,\tau+1} 1_{\mathcal{S}\underset{\mathrm{s}}{\sim}\{\tilde\tau,\tau'\}} \right) \prod_{S\in\mathcal{S}} w_{\mathrm{s}}^{(1+\alpha)}(S) \\
&\leq 3C
\end{aligned}
\tag{2.91}
$$

where we have picked an arbitrary τ' and $\{\tilde\tau, \tau'\}$ is meant as an edge E and not as an interval J (also in the formulas below).

This yields the bound in item 1) of the Lemma. The translation invariance property in item 1) follows because the translation invariance in item 1 of Proposition 2.5, gives rise to the property

$$
w_{\mathrm{s}}^{(\beta)}(S) = w_{\mathrm{s}}^{(\beta)}(S+\tau),
\tag{2.92}
$$

where $S+\tau$ is the shifted set with the same label as S. To deal with item 2), i.e. the case $0 \in A$, we replace (2.89) (more precisely, its sum over A) by

$$
\sum_{A:0\in A} \mathrm{e}^{c|A|} d(A)^{1+\alpha} |v(A)| \leq \sum_{\tau\geq 1} \langle\tau\rangle^{1+\alpha} \hat{e}(0,\tau) \sum_{\mathcal{S}\subset\mathscr{S}} 1_{\mathcal{S}\cup\{\{0,\tau\}\}\,\text{connected}} \prod_{S\in\mathcal{S}} w_{\mathrm{s}}^{(1+\alpha)}(S)
\tag{2.93}
$$

$$
+ \sum_{J:0\in J} C|J|^{1+\alpha} \mathrm{e}^{-(g-c)|J|} \lambda^2 \sum_{\mathcal{S}\subset\mathscr{S}} 1_{\mathcal{S}\cup\{J\}\,\text{connected}} \prod_{S\in\mathcal{S}} w_{\mathrm{s}}^{(1+\alpha)}(S)
\tag{2.94}
$$

The two terms correspond to the fusions where $0 \in \mathrm{Supp}\mathcal{A}$, $0 \in \mathrm{Supp}\mathcal{J}$, respectively. Note that in the second term we know that there is at least one $S \in \mathscr{S}$ that is an edge $E = \{\tau, \tau'\}$ and therefore we can extract a factor λ^2. In the first term, this might not be the case. Apart from this, the verification of (2.94) goes as in (2.89). In both terms, we estimate the sum over $\mathcal{S}$ by C, as above. Then, the remaining sum over τ yields a factor $\breve{C}|\lambda|$ by (2.75), and the sum over intervals J yields $C|\lambda|^2$. We thus obtain (2.80).

Next, we turn to the case where $n+1 \in A$. Let us add to the set $\mathscr{S}$ also self-edges $E = \{\tau_0\}$, keeping the same definition (2.87) for the relation $\underset{\mathrm{s}}{\sim}$ (but we do not need to assign a weight $w^{(\beta)}$ to self-edges). Let us again for simplicity assume $0 \notin A$. Analogously to (2.89), we derive

$$
\mathrm{e}^{c|A|} |v(A\cup\{n+1\})| \leq C \sum_{\tau_0,\tau_0'\in A, \tau_0\leq\tau_0'} \hat{e}(\tau_0,\tau_0'|b) \sum_{\mathcal{S}\subset\mathscr{S}} 1_{\mathrm{Supp}\mathcal{S}'=A} 1_{\mathcal{S}'\,\text{connected}} \prod_{S\in\mathcal{S}} w_{\mathrm{s}}^{(0)}(S),
\tag{2.95}
$$

with $\mathcal{S}' = \mathcal{S} \cup \{E = (\tau_0, \tau_0')\}$. Note that we did not extract a factor λ^2 from the RHS, in contrast to (2.89). Indeed, in the case where $\mathcal{A}$ contains only one set A, and this A is a singleton, no such small factor is present because there is no small factor in (2.77). Note furthermore that (2.95) of course remains valid when we multiply the LHS by $d(A)^\beta$, and, on the RHS, we replace $\hat{e}(\tau_0, \tau_0'|b)$ by $\langle\tau_0' - \tau_0\rangle^\beta \hat{e}(\tau_0, \tau_0'|b)$ and $w_{\mathrm{s}}^{(0)}$ by $w_{\mathrm{s}}^{(\beta)}$, for $\beta \leq 1+\alpha$. To obtain Item 4) of the lemma, we use (2.95) with these replacements, choosing $\beta = 1+\alpha$. Using the same strategy

as in the proof of item 1), we first sum over the collections $\mathcal{S}$ (note that this collection does not contain the pair $(\tau_0, \tau_0'))$ and then the bound (2.77) for the edge factor $\hat{e}(\tau_0, \tau_0'|b)$.

Item 3) is also derived as above, choosing now $\beta = 1 + \alpha/2$; the only difference is that we can extract an additional small factor $\delta^{\alpha/2}$ from the edge factor $\hat{e}(\tau, \tau'|b)$, i.e. we use (2.76).

Finally, we deal with item 5), starting from the estimate (2.78). We do this in a more abstract way than necessary, because at a later stage we will have to perform a very similar calculation. We recast (2.95) as

$$\mathrm{e}^{c|A|}|v(A \cup \{n+1\})| \leq C \sum_{A_0 \subset A, 1 \leq |A_0| \leq 2} |x(A_0)||z_{A_0}(A \setminus A_0)| \tag{2.96}$$

where we have set $A_0 = \{\tau_0, \tau_0'\}$, possibly with $\tau_0 = \tau_0'$, and

$$x(A_0) := \hat{e}(\tau_0, \tau_0'|b), \tag{2.97}$$

$$z_{A_0}(A_1) := \sum_{\mathcal{S} \subset \mathscr{S}} 1_{\mathrm{Supp}\mathcal{S} = A_1} 1_{\mathcal{S}' \text{ connected}} \prod_{S \in \mathcal{S}} w_{\mathrm{s}}^{(0)} \tag{2.98}$$

with, as above, $\mathcal{S}' = \mathcal{S} \cup \{\{\tau_0, \tau_0'\}\}$, and $z_{A_0}(\emptyset) := 1$. We check that $z_{A_0}(A_1)$ satisfies the following properties

a) $z_{A_0}(A_1) = 0$ unless $\mathrm{dist}(A_0, A_1) \leq 1$. This connectedness property is inherited from the connectedness of $\mathcal{S}'$ in (2.98).

b) By similar reasoning as in (2.91), we get (with $|a|_+ = \max(a, 0)$ for $a \in \mathbb{R}$)

$$\sum_{A_1} \langle |\min A_0 - \min A_1|_+ \rangle^{1+\alpha} |z_{A_0}(A_1)| \leq C. \tag{2.99}$$

One could also multiply the LHS with, say, $\langle |\max A_1 - \max A_0|_+ \rangle^{1+\alpha}$ but the above will suffice for our purposes.

c) A slight variation of *b)*:

$$\sum_{\tau > 0} \langle \tau \rangle^{1+\alpha} \sup_{\tau_1, \tau_0 : \tau = \tau_0 - \tau_1} \sup_{A_0 : \min A_0 = \tau_0} \sum_{A_1 : \min A_1 = \tau_1} |z_{A_0}(A_1)| \leq C. \tag{2.100}$$

Note the translation invariance in this bound (i.e. the fact that the sum over τ can be estimated regardless of τ_0, τ_1. This is of course a consequence of the translation invariance (2.92).

To prove item 5), we have to estimate

$$\sum_{A_0, A_1, \min(A_0 \cup A_1) \leq \tilde{n}} \langle \tilde{n} - \min(A_0 \cup A_1) \rangle^{\alpha} |x(A_0)||z_{A_0}(A_1)| \tag{2.101}$$

where we suppress $A_0, A_1 \in I_{1,n}$ and $1 \leq |A_0| \leq 2$ in the notation. First, restrict this sum to the case where $\min A_0 \leq \tilde{n}$, such that, since $\mathrm{dist}(A_0, A_1) \leq 1$,

$$\langle \tilde{n} - \min(A_0 \cup A_1) \rangle \leq C \langle \tilde{n} - \min A_0 \rangle \langle |\min A_0 - \min A_1|_+ \rangle \tag{2.102}$$

Therefore, we get a bound for (2.101) by first performing the sum over A_1 using item b) above, and obtaining a constant C, and then summing over A_0 using (2.78) (which one could restate as a property of the x factors).

Now we consider the case $\min A_1 \leq \tilde{n} < \min A_0$, where we have

$$\langle \tilde{n} - \min(A_0 \cup A_1) \rangle = \langle \tilde{n} - \min A_1 \rangle \tag{2.103}$$

so our task boils down to bounding

$$\sum_{A_0, A_1 : \min A_1 \leq \tilde{n} < \min A_0} \langle \tilde{n} - \min A_1 \rangle^\alpha |z_{A_0}(A_1)||x(A_0)| \tag{2.104}$$

$$\leq \sum_{\tau_{0,1} : \tau_1 \leq \tilde{n} < \tau_0} \langle \tau_0 - \tau_1 \rangle^\alpha \left(\sup_{A_0 : \min A_0 = \tau_0} \sum_{A_1 : \min A_1 = \tau_1} |z_{A_0}(A_1)| \right) \left(\sum_{A_0 : \min A_0 = \tau_0} |x(A_0)| \right) \tag{2.105}$$

$$\leq \breve{C} \sum_{\tau \geq 1} \langle \tau \rangle^{1+\alpha} \sup_{\tau_1, \tau_0 : \tau = \tau_0 - \tau_1} \left(\sup_{A_0 : \min A_0 = \tau_0} \sum_{A_1 : \min A_1 = \tau_1} |z_{A_0}(A_1)| \right) \leq C \tag{2.106}$$

The first inequality is just a $l^1 - l^\infty$ inequality. To get the second, we bound the sum over A_0 with τ_0 fixed by $\breve{C}$ (independent of τ_0) by (2.77) and then we perform the sum over τ_0, τ_1 with $\tau = \tau_1 - \tau_0$ fixed and with $\tau_1 \leq \tilde{n} < \tau_0$; this sum has τ terms. The last inequality is item c) above.

3 Proofs of the main theorems

In this Section, we give the final proof of our main results, Theorems 1.4 and 1.1. First, in Section 3.1, we introduce some general tools, applying to both choices of the operator b. Most importantly, we develop a refinement of the representation (2.42). In Section 3.2, we specialize to the case $b = \theta(x/t_c)$ and we prove the propagation bound, i.e. Theorem 1.4. In Section 3.3, we take $b = \theta(k/\delta)$ and we obtain the soft photon bound, i.e. Theorem 1.1.

3.1 General considerations

As announced, we do not distinguish for now between the two different choices for b, except in Lemma 3.1. We start from the representation (2.42) and we introduce some notation to simplify it. We will use the adjacency relation $A \sim A' \Leftrightarrow \mathrm{dist}(A, A') \leq 1$ for subsets of $I_{0,n}$, and extended to subsets of $I_{0,n+1}$ by simply ignoring the element $n+1$, i.e.:

$$A \sim A' \Leftrightarrow \mathrm{dist}(A \setminus \{n+1\}, A' \setminus \{n+1\}) \leq 1, \qquad A, A' \neq \{n+1\} \tag{3.1}$$

(we never need the case where A or A' is the singleton $\{n+1\}$). As previously, we write $\mathcal{A} \sim A'$ if there is at least one $A \in \mathcal{A}$ such that $A \sim A'$, and $\mathcal{A} \nsim A'$ if there is no $A \in \mathcal{A}$ such that $A \sim A'$.

We recast (2.42) by separating each collection $\mathcal{A}$ into its boundary and bulk polymers;

$$Z_n = \sum_{A_\rtimes} v(A_\rtimes) Z_{n, A_\rtimes} + \sum_{A_\rtimes, A_\ltimes : A_\rtimes \nsim A_\ltimes} v(A_\ltimes) v(A_\rtimes) Z_{n, A_\rtimes \cup A_\ltimes} + \sum_{A_{\ltimes,\rtimes}} v(A_{\ltimes,\rtimes}) Z_{A_{\ltimes,\rtimes}} \tag{3.2}$$

where $A_\ltimes, A_\rtimes, A_{\ltimes,\rtimes}$ run over nonempty subsets of $I_{0,n+1}$ that, respectively,

- contain 0 but not $n+1$,
- contain $n+1$ and at least one other element, but not 0.
- contain both 0 and $n+1$.

and the factors $Z_{n,A}$ in (3.2) are defined as

$$Z_{n,A'} := \sum_{\substack{\mathcal{A} \in \mathfrak{B}^1_{1,n} \\ \mathcal{A} \nsim A'}} \prod_{A \in \mathcal{A}} v(A) \tag{3.3}$$

Note that $Z_{n,A}$ depends only on bulk polymer weights. Moreover, by (2.39),

$$1 = Z_n(\mathbb{1}, \rho_0) = Z_{n,\emptyset}, \qquad \text{for } \rho_0 = \eta \otimes P_\Omega \tag{3.4}$$

As explained in [1], the quantity $Z_{n,A'}$ can be viewed as the partition function of a polymer gas with polymer weights $w(A) \equiv v(A) 1_{[A \nsim A']}$. For λ small enough, the bound (2.79)(a 'Kotecky-Preiss' criterion', in the terminology of [1]) allows us to apply the cluster expansion and obtain

$$\log Z_{n,A'} = \sum_{\mathcal{A} \in \mathfrak{B}_{1,n}} v^T(\mathcal{A}) 1_{[\mathcal{A} \nsim A']} \tag{3.5}$$

where the *truncated* weights $v^T(\cdot)$ are defined starting from the weights $v(\cdot)$, see Appendix A of [1]. The only properties of the weights $v^T(\cdot)$ that we need are[3]

$$\sum_{\mathcal{A} \in \mathfrak{B}_{1,n}, \mathcal{A} \sim A} d(\mathcal{A})^{1+\alpha} \mathrm{e}^{c|\mathrm{Supp}\mathcal{A}|} |v^T(\mathcal{A})| \leq C|\lambda^2||A|, \tag{3.6}$$

and the translation-invariance, from Lemma 2.8 item 1.

$$v^T(\mathcal{A}) = v^T(\mathcal{A} + \tau), \qquad \text{with } \mathcal{A} + \tau := \{A + \tau, A \in \mathcal{A}\}, \tag{3.7}$$

for τ such that $\mathcal{A} + \tau \in \mathfrak{B}_{1,n}$. Comparing to the expansion of $\log Z_{n,\emptyset} = 0$, we get

$$\log Z_{n,A'} = \log \frac{Z_{n,A'}}{Z_{n,\emptyset}} = - \sum_{\mathcal{A} \in \mathfrak{B}_{1,n}} v^T(\mathcal{A}) 1_{[\mathcal{A} \sim A']} \tag{3.8}$$

Relying on (3.8), we can simplify (3.2) by modifying the weights $v(\cdot)$ into new weights $\bar{v}(\cdot)$, obtaining

$$Z_n = \sum_{A_\rtimes} \bar{v}(A_\rtimes) + \sum_{A_\ltimes, A_\rtimes : A_\ltimes \nsim A_\rtimes} \bar{v}(A_\ltimes) \bar{v}(A_\rtimes) + \sum_{A_{\ltimes,\rtimes}} \bar{v}(A_{\ltimes,\rtimes}) \tag{3.9}$$

where the symbols $A_\ltimes, A_\rtimes, A_{\ltimes,\rtimes}$ have the same meaning as before in (3.2) and the wieghts $\bar{v}(\cdot)$ are defined (for sufficiently small λ) as follows, we set

$$\bar{v}(A) = \bar{v}^{(1)}(A) + \bar{v}^{(2)}(A) \tag{3.10}$$

[3] The property stated in Appendix A of [1] misses the factor $\mathrm{e}^{c|\mathrm{Supp}\mathcal{A}|}$ but this can be easily obtained by redefining $v(A) \to \mathrm{e}^{c|A|} \cdot v(A)$

where

$$\bar{v}^{(1)}(A) := \sum_{\substack{A_0 \subset I_{0,n+1}, \mathfrak{A} \subset \mathcal{B}_{1,n} \\ A_0 \cup \mathrm{Supp}\mathfrak{A} = A}} v(A_0) \prod_{\mathcal{A} \in \mathfrak{A}} (\mathrm{e}^{-v^T(\mathcal{A})} - 1) 1_{[\mathcal{A} \sim A_0]} \tag{3.11}$$

$$\bar{v}^{(2)}(A) := \sum_{\substack{A_\ltimes, A_\rtimes, A_\ltimes \nsim A_\rtimes, \mathfrak{A} \subset \mathcal{B}_{1,n} \\ A_\ltimes \cup A_\rtimes \cup \mathrm{Supp}\mathfrak{A} = A}} v(A_\ltimes) v(A_\rtimes) \prod_{\mathcal{A} \in \mathfrak{A}} (\mathrm{e}^{-v^T(\mathcal{A})} - 1) 1_{[\mathcal{A} \sim A_\ltimes]} 1_{[\mathcal{A} \sim A_\rtimes]} \tag{3.12}$$

with $\prod_{\mathcal{A}}$ is set 1 if $\mathfrak{A} = \emptyset$ and $\mathrm{Supp}\mathfrak{A} = \cup_{\mathcal{A} \in \mathfrak{A}} \mathrm{Supp}\mathcal{A}$. We note that the term $v^{(2)}$ is absent unless A contains both 0 and $n+1$.

The weights $\bar{v}(\cdot)$ are similar to the weights $v(\cdot)$, because the $v^T(\cdot)$ are small and they have strong summability properties, as displayed in (3.6). More precisely, we get

Lemma 3.1. *Items $2, 3, 4, 5$ of Lemma 2.8 hold with $v(\cdot)$ replaced by $\bar{v}(\cdot)$, possibly with different constants $c, C, \breve{C}$. We will henceforth refer to items $2, 3, 4, 5$ of Lemma 3.1.*

We will prove item 2) and we sketch the proof of item 5). The proof of the other items is analogous. More generally, the proof of Lemma 3.1 is in spirit very similar to the proof of Lemma 2.8 itself, using Lemma 2.8 for the $v(\cdot)$ weights and the bound (3.6) for the $v^T(\cdot)$ weights.

Proof of item 2). Since this item deals with A containing 0 but not $n+1$, we have $\bar{v} = \bar{v}^{(1)}$. We rewrite the sum over $\mathfrak{A} \subset \mathfrak{B}_{1,n}$ in (3.11) by first summing over $k = |\mathfrak{A}|$, and, for each $\mathcal{A} \in \mathfrak{A}$, over $\tau_{\mathcal{A}}$, defined as the smallest element of A_0 such that $\{\tau_{\mathcal{A}}\} \sim \mathcal{A}$. Then, we order the times $\tau_{\mathcal{A}}$ and rename them as $\tau_1, \dots, \tau_k$. This yields

$$\sum_{A:0 \in A} |\bar{v}(A)| \mathrm{e}^{c|A|} d(A)^{1+\alpha} \leq \sum_{A_0:0 \in A_0} |v(A_0)| \mathrm{e}^{c|A_0|} d(A_0)^{1+\alpha} \tag{3.13}$$

$$\sum_{k=1}^{\infty} \sum_{\substack{\tau_1, \dots, \tau_k \in A_0 \\ \tau_1 \leq \tau_2 \leq \dots \leq \tau_k}} \prod_{j=1}^{k} \sum_{\substack{\mathcal{A} \in \mathfrak{B}_{1,n} \\ \mathcal{A} \sim \{\tau_j\}}} |\mathrm{e}^{v^T(\mathcal{A})} - 1| d(\mathcal{A})^{1+\alpha} \mathrm{e}^{c|\mathrm{Supp}\mathcal{A}|}$$

We bound the sums over $\mathcal{A}$ by $C|\lambda|^2$ by using (3.6) with $A = \{\tau_j\}$, then the sum over $\tau_1, \dots, \tau_k$ yields $(C\lambda^2|A_0|)^k/k!$. The sum over k yields an exponential and therefore, (3.13) is bounded by

$$\sum_{A_0:0 \in A} |v(A_0)| \mathrm{e}^{(c+C\lambda^2)|A_0|} d(A_0)^{1+\alpha} \tag{3.14}$$

By reducing c and $|\lambda|$, we can make $c' := (c + C\lambda^2)$ sufficiently small such that item 2) of Lemma 2.8 holds with c' in the role of c. Then, (3.14) is bounded by $|\lambda|\breve{C}$ by item 2) of Lemma 2.8 and item 2) of Lemma 3.1 is proven. □

Sketch of Proof of item 5). We restrict ourselves to the case where $0 \notin A$ (the other case is easier), such that again $\bar{v} = \bar{v}^{(1)}$. Analogous to (2.96), we can represent

$$\bar{v}(A) = \sum_{A_0 \subset A} z_{A_0}(A \setminus A_0) x(A_0) \tag{3.15}$$

with

$$x(A_0) := v(A_0 \cup \{n+1\}) \tag{3.16}$$

$$z_{A_0}(A_1) := \sum_{\substack{\mathfrak{A} \subset \mathcal{B}_{1,n} \\ \cup_{\mathcal{A} \in \mathfrak{A}} \mathrm{Supp}\mathcal{A} = A_1}} \prod_{\mathcal{A} \in \mathfrak{A}} (\mathrm{e}^{-v^T(\mathcal{A})} - 1) 1_{\mathcal{A} \sim A_0} \tag{3.17}$$

(again we set $z_{A_0}(\emptyset) = 1$) and with the properties (cfr. the properties a, b, c following (2.96))

$a)$ $z_{A_0}(A_1) = 0$ unless $A_0 \sim A_1$ (trivial from (3.17))

$b)$ From (3.6), we obtain

$$\sum_{A_1 \subset I_{1,n}, A_1 \neq \emptyset} \langle | \min A_0 - \min A_1 |_+ \rangle^{1+\alpha} |z_{A_0}(A_1)| \leq \mathrm{e}^{C|\lambda|^2 |A_0|} - 1 \tag{3.18}$$

by proceeding as in the proof of item 2) above.

$c)$ A slight variation of $b)$:

$$\sum_{\tau > 0} \langle \tau \rangle^{1+\alpha} \sup_{\tau_1, \tau_0 : \tau = \tau_0 - \tau_1} \sup_{A_0 : \min A_0 = \tau_0} \mathrm{e}^{-C\lambda^2 |A_0|} \sum_{A_1 : \min A_1 = \tau_1} |z_{A_0}(A_1)| \leq C|\lambda|^2 \tag{3.19}$$

which relies (apart from the considerations in the proof of item 2)) on the invariance property (3.7).

To get item 5) (with $0 \notin A$) we control (cfr. (2.101))

$$\sum_{A_0, A_1, \min(A_0 \cup A_1) \leq \tilde{n}} \langle \tilde{n} - \min(A_0 \cup A_1) \rangle^{\alpha} \mathrm{e}^{c|A_1|} |z_{A_0}(A_1)| \mathrm{e}^{c|A_0|} |x(A_0)| \tag{3.20}$$

with A_0, A_1 running over subsets of $I_{1,n}$ and $\tilde{n} = n - m_\theta n_c$ The proof now proceeds as the proof of item 5) of Lemma 2.8, based on the representation (3.15). The only difference is that now possibly $|A_0| > 2$, which results in factors $\mathrm{e}^{C\lambda^2 |A_0|}$. As in the proof of item 2) above, this is handled by choosing the constant c sufficiently small. Note also that, whereas in the proof of item 5) of Lemma 2.8, we had estimates on $x(A_0)$ because of bounds on the $\hat{e}(\tau_0, \tau_0' | b)$, in the present proof, the analogous estimates on $x(A_0)$ are provided by Items 4),5) of Lemma 2.8.

□

Finally, we rewrite (3.9) by first remarking that (for any n)

$$\sum_{A_\ltimes} \bar{v}(A_\ltimes) = 0 \tag{3.21}$$

Indeed, consider (3.9) for $Z_n(\mathbb{1}, \rho_0) = 1$, then polymers A with $n+1 \in A$ never appear and in that case (3.9) simply reads $1 = 1 + \sum_{A_\ltimes} \bar{v}(A_\ltimes)$. Then, we decompose $\sum_{A_\ltimes \nsim A_\rtimes} = (\sum_{A_\ltimes})(\sum_{A_\rtimes}) - \sum_{A_\ltimes \sim A_\rtimes}$ so that we get

$$Z_n = \sum_{A_\rtimes} \bar{v}(A_\rtimes) + \sum_{A_\ltimes \sim A_\rtimes} \bar{v}(A_\ltimes) \bar{v}(A_\rtimes) + \sum_{A_{\ltimes,\rtimes}} \bar{v}(A_{\ltimes,\rtimes}) \tag{3.22}$$

3.2 Propagation bound

In this section, we specialize to $b = \theta(x/t_c)$ and we again abbreviate $Z_n = Z_n(\mathrm{d}\Gamma(b), \rho_0)$.

We start from the expression (3.9), and we first establish

Lemma 3.2.

$$\left| Z_n - \sum_{A_{\rtimes}} \bar{v}(A_{\rtimes}) \right| \leq \breve{C} \langle n \rangle^{-\alpha} \tag{3.23}$$

Proof. From (3.22), the expression between $|\cdot|$ equals

$$- \sum_{A_{\ltimes} \sim A_{\rtimes}} \bar{v}(A_{\ltimes}) \bar{v}(A_{\rtimes}) + \sum_{A_{\ltimes,\rtimes}} \bar{v}(A_{\ltimes,\rtimes}) \tag{3.24}$$

Let us treat the first term. We distinguish the cases $\min A_{\rtimes} \leq (n - m_\theta n_c)/2$ and $\min A_{\rtimes} > (n - m_\theta n_c)/2$. In the first case, we first sum over $A_{\ltimes}$ using item 2 of Lemma 3.1, yielding a factor $\breve{C}$ (in fact $\breve{C}|\lambda|$) and then over $A_{\rtimes}$, using item 5 of Lemma 3.1 and the fact that

$$n - n_c m_\theta - (n - m_\theta n_c)/2 \geq (1 - m_\theta) n/2 = \breve{c} n,$$

and obtaining $\breve{C}\langle n \rangle^{-\alpha}$ (recall that $n_c \leq n$). In the second case, we first sum over $A_{\ltimes}$ (item 2 of Lemma 3.1) obtaining a factor $\breve{C}\langle \min A_{\rtimes} \rangle^{-(1+\alpha)}$ because $\max A_{\ltimes} \geq \min A_{\rtimes} - 1$ (since $A_{\ltimes} \sim A_{\rtimes}$), then we sum over $A_{\rtimes}$ keeping $\min A_{\rtimes}$ fixed, yielding $\breve{C}$ by item 4 of Lemma 3.1, and finally over $\min A_{\rtimes}$, yielding $\breve{C}\langle (n - m_\theta n_c)/2 \rangle^{-\alpha} = \breve{C}\langle n \rangle^{-\alpha}$. The second term in (3.24) is estimated in an analogous way. □

Now we are ready to consider the limit $n \to \infty$ in the expression for $Z_n(\mathrm{d}\Gamma(b), \rho_0)$. Up to now, we did not indicate the final time n explicitly in our notation for the polymer weights $v(\cdot), \bar{v}(\cdot)$ but for the purposes of this section, it is advantagous to do so, as we will be comparing different final times n, n', and because these weights do depend on the final time. Therefore, let us write temporarily

$$\bar{v}_n(A) \quad \text{instead of} \quad \bar{v}(A) \tag{3.25}$$

and note that, whereas $v_n(A)$ for $A \ni n+1$ is not invariant under shifting the set A if $b = \theta(x/t_c)$, we can recover invariance by shifting A together with n, but keeping t_c (hence n_c) fixed, i.e.

$$\bar{v}_n(A) = \bar{v}_{n+\tau}(A + \tau), \qquad \tau \in \mathbb{N}, \tag{3.26}$$

as we easily derive from item 1) of Proposition 2.5. Therefore, it natural to introduce a new notation, namely

$$\nu(A) := \bar{v}_n(n + 1 - A), \qquad A \subset I_{0,n}, 0 \in A, \tag{3.27}$$

where the condition $n \geq \max A$ ensures that $n + 1 - A \in I_{1,n+1}$ (recall that the sum over $A_{\rtimes}$ runs over sets not including 0). With this notation, we have

$$\sum_{A_{\rtimes}} \bar{v}(A_{\rtimes}) = \sum_{A \subset I_{0,n}, \min A = 0} \nu(A) \tag{3.28}$$

Let us define

$$Z_\infty := \sum_{A\in\mathbb{N},\min A=0} \nu(A) \tag{3.29}$$

Note that Z_∞ still depends on t_c (or n_c), but of course not on ρ_0.

Lemma 3.3. *This sum on the RHS of* (3.29) *is absolutely convergent and*

$$\left| Z_\infty - \sum_{A\subset I_{0,n},\min A=0} \nu(A) \right| \leq \breve{C}\langle n\rangle^{-\alpha} \qquad , \tag{3.30}$$

$$|Z_n - Z_\infty| \leq \breve{C}\langle n\rangle^{-\alpha} \tag{3.31}$$

Proof. The first bound follows by using items 4) and 5) of Lemma 3.1 and proceeding similarly (though simpler) as in the proof of Lemma 3.2. To get the second bound, we substitute the rewriting (3.28) into (3.23) and use (3.30). □

3.2.1 Proof of Theorem 1.4

First, we slightly generalize the setup used in this paper, and we consider $Z_n(O,\rho_0)$ with $O = \mathrm{d}\Gamma(b), \mathbb{1}$ and ρ_0 now not longer a density matrix but the rank-1 operator

$$\rho_0 = \big|\psi_{\mathrm{S}} \otimes \mathcal{W}(\psi_\ltimes)\Omega\big\rangle\big\langle\psi'_{\mathrm{S}} \otimes \mathcal{W}(\psi'_\ltimes)\Omega\big| \tag{3.32}$$

with $\psi_{\mathrm{S}}, \psi'_{\mathrm{S}} \in \mathscr{H}_{\mathrm{S}}$ and $\psi_\ltimes, \psi'_\ltimes \in \mathfrak{h}_\alpha$. We can easily go through all arguments, with obvious changes, and get the following analogue of Lemma 3.3

$$|Z_n(\mathrm{d}\Gamma(b),\rho_0) - Z_\infty \operatorname{Tr}\rho_0| \leq \breve{C}\langle n\rangle^{-\alpha}, \qquad Z_n(\mathbb{1},\rho_0) = \operatorname{Tr}\rho_0 \tag{3.33}$$

with Z_∞ as above in (3.29). Now, take $\Psi \in \mathcal{D}_\alpha$, i.e. a finite linear combination $\Psi = \sum_i \Psi_i$ with $\Psi_i = \psi_{\mathrm{S},i} \otimes \mathcal{W}(\psi_{\ltimes,i})\Omega$, then

$$\lim_{n\to\infty} \langle \Psi(n/\lambda^2), \mathrm{d}\Gamma(b)\Psi(n/\lambda^2)\rangle = Z_\infty \sum_{i,j} \langle \Psi_i, \Psi_j\rangle = Z_\infty \|\Psi\|^2 \tag{3.34}$$

Hence, we get the statement of Theorem 1.4 for times t taken along a subsequence n/λ^2 (and t_c of the form n_c/λ^2). To get the full statement, we should again generalize the reasoning in a straightforward way.

Assume that the time-discretization of the model was chosen based on 'mesoscopic time-blocks' of length $\ell|\lambda|^{-2}, \ell \in [1,2]$, instead of $\ell = 1$ as we did previously: this means that we change the definition of $Q_n, Q_{b,n}$ and $U_\tau, \tau = 1,\ldots,n$ by replacing $|\lambda|^{-2}$ by $\ell|\lambda|^{-2}$, for example, instead of (2.6), we have

$$U_\tau := \mathrm{e}^{\mathrm{i}(\tau\ell/\lambda^2)L_{\mathrm{F}}}\mathrm{e}^{-\mathrm{i}(\ell/\lambda^2)L}\mathrm{e}^{-\mathrm{i}(\tau-1)\ell/\lambda^2 L_{\mathrm{F}}}, \qquad \tau \in I_{1,n} \tag{3.35}$$

Then, Lemma 2.1 holds as well with a constant $C^{(\ell)}$ and gap $g^{(\ell)}$ that can be chosen uniform in $\ell \in [1,2]$, as we easily get from the results in [1], in particular from the proof of Lemma 2.3 1) therein. The rest of the reasoning goes through without any change except for the readjusting of constants. Hence we have proven Theorem 1.4 restricted to times t taken along

a subsequence $n\ell/\lambda^2$ with a constant $\breve{C}$ on the RHS that can be chosen uniform in $\ell \in [1,2]$. Finally, t_c can be tuned independently of t by changing the function $\theta(\cdot)$ to $\theta(\cdot/\ell)$ for $\ell \in [1,2]$ and again the constants $\breve{C}$ can be chosen uniform. This indeed allows to choose any $t_c \geq \lambda^{-2}$ (smaller t_c would require to take ℓ dependent on λ which we prefer to avoid). This of course establishes the full Theorem 1.4.

3.3 Soft photon bound

In this section, we take to $b = \theta(k/\delta)$. We start from (3.22) that we recall for convenience

$$Z_n = \sum_{A_{\rtimes}} \bar{v}(A_{\rtimes}) - \sum_{A_{\ltimes} \sim A_{\rtimes}} \bar{v}(A_{\ltimes})\bar{v}(A_{\rtimes}) + \sum_{A_{\ltimes,\rtimes}} \bar{v}(A_{\ltimes,\rtimes}) \tag{3.36}$$

The second and third term on the RHS can be estimated by $\delta^{\alpha/2}\breve{C}$ by using items 2) and 3) of Lemma 3.1. For the first term on the RHS, we argue

Lemma 3.4. *There is an n-independent number a such that*

$$\left| \sum_{A_{\rtimes}} \bar{v}(A_{\rtimes}) - na \right| \leq \breve{C}\delta^{\alpha/2} \tag{3.37}$$

Proof. Let us again make the dependence on the final time n explicit by writing $\bar{v}_n(A)$ instead of $\bar{v}(A)$, as in Section 3.2. By the independence properties of item 1 of Lemma 2.8, we get, for A not containing $n+1$,

$$\bar{v}_n(A \cup \{n+1\}) = \bar{v}_{n+\tau}(A \cup \{n+\tau+1\}) \tag{3.38}$$

(note that this is a stronger invariance property than (3.26)). Therefore, it makes sense to introduce the new notation

$$\bar{v}(A|b) := \lim_{n\to\infty} \bar{v}_n((A \cup \{n+1\}), \tag{3.39}$$

Then, let us define

$$a := \sum_{A \subset \mathbb{N}_0, \min A = 1} \bar{v}(A|b) \tag{3.40}$$

where the sum is absolutely convergent by item 3 of Lemma 3.1, and we have

$$na - \sum_{A \subset I_{1,n}} \bar{v}(A|b) = \sum_{A \subset \mathbb{N}, \min A = 1} \min(\max A - 1, n)\bar{v}(A|b) \tag{3.41}$$

The LHS is the expression between $|\cdot|$ in (3.37), and the RHS can be bounded by $\breve{C}\delta^{\alpha/2}$ by using again item 3 of Lemma 3.1. ☐

By the photon bound in [1], we know that $\sup_t \langle \Psi_t, N\Psi_t \rangle \leq \breve{C}$, and therefore $\sup_n Z_n(\mathrm{d}\Gamma(b), \rho_0) \leq \breve{C}$, see the remark following Theorem 1.1 . However, Lemma 3.4 and the bounds on the other

terms (second and third) of (3.36) imply that $Z_n(\mathrm{d}\Gamma(b), \rho_0) - an$ is uniformly bounded in n. Combining these two statements, we conclude $a = 0$, and therefore, we have shown

$$|Z_n(\mathrm{d}\Gamma(b), \rho_0)| \leq \breve{C}\delta^{\alpha/2} \tag{3.42}$$

We have hence obtained Theorem 1.1 for t restricted to particular vectors $\Psi_0 = \psi_{\mathrm{S},0} \otimes \mathcal{W}(\psi_{\ltimes})\Omega$ and times of the form $t = n/\lambda^2$. By the same trick as applied at the end of the proof of Theorem 1.4 in Section 3.2.1 (involving the change of mesoscopic scale $|\lambda|^{-2} \to \ell|\lambda|^{-2}$), we get the statement for all times t. By the Cauchy-Schwarz inequality, we get the statement for any $\Psi \in \mathcal{D}_\alpha$. This proves the full Theorem 1.1.

A One-particle estimates

In this appendix we prove some estimates concerning the dynamics of a single free photon. Very similar estimates were also established by different methods in [3] (the approach here is less elegant, but more self-contained)

Since this section stands apart from the rest of the paper, we chose to make it self-contained apart from a single definition below. In particular, we do not adhere to our earlier convention to distinguish constants C and $\breve{C}$. First, we state the auxiliary

Lemma A.1. *For $f \in L^2(\mathbb{R}^d)$, assume that $\hat{f} \in C^1(\mathbb{R}^d \setminus \{0\})$ such that for some $0 < \gamma < 1$,*

$$|k|^{1-\gamma}\partial\hat{f} \in L^1(\mathbb{R}^d; \mathbb{C}^d), \qquad |k|^{-\gamma}\hat{f} \in L^1(\mathbb{R}^d) \tag{A-1}$$

Then,

$$|f(x)| \leq C(\gamma)|x|^{-\gamma}\left(\||k|^{-\gamma}\hat{f}\|_1 + \||k|^{1-\gamma}\partial\hat{f}\|_1\right)$$

Proof. We write

$$f(x) = \frac{1}{\mathrm{e}^{-\mathrm{i}} - 1}\int (\hat{f}(k + \hat{x}/|x|) - \hat{f}(k))\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k.$$

Divide the integral to $|k| \leq 2|x|^{-1}$ and $|k| > 2|x|^{-1}$. For the first one insert $1 \leq 2^\gamma|x|^{-\gamma}|k|^{-\gamma}$ and the integral is bounded by $2^{1+\gamma}|x|^{-\gamma}\||k|^{-\gamma}\hat{f}\|_1$. For the second integral, we can assume that $\hat{f}$ is C^1, hence we insert

$$\hat{f}(k + \hat{x}/|x|) - \hat{f}(k) = |x|^{-1}\int_0^1 \mathrm{d}s\, \hat{x} \cdot \partial\hat{f}(k + s\hat{x}/|x|).$$

to bound it by

$$\begin{aligned}\int_0^1 \mathrm{d}s \int_{|k|>\frac{2}{|x|}} \mathrm{d}k\, |x|^{-1}|\partial\hat{f}(k + s\hat{x}/|x|)| &\leq |x|^{-\gamma}\int_{|k|>\frac{1}{|x|}} \mathrm{d}k\, |x|^{-1+\gamma}|\partial\hat{f}(k)| \\ &\leq C|x|^{-\gamma}\||k|^{1-\gamma}\partial\hat{f}\|_1 \end{aligned} \tag{A-2}$$

since $|x|^{-1+\gamma} \leq |k|^{1-\gamma}$ in the last integral. □

A.1 Propagation estimate

Recall the function θ introduced above Theorem 1.1 . It is a spherically symmetric C^∞ function $\mathbb{R}^d \to [0,1]$ with compact support such that $\sup|\mathrm{Supp}\theta| < 1$ and we write $\theta_t(x) := \theta(x/t)$. Recall also the dense subspace $\mathfrak{h}_\alpha \subset L^2(\mathbb{R}^d)$. We prove

Lemma A.2. *Let $d \geq 3$ and $\psi, \psi' \in \mathfrak{h}_\alpha$. Pick m_θ such that $1 > m_\theta > \sup|\mathrm{Supp}\theta|$, then*

$$\int_u^\infty \mathrm{d}v \langle t_2 - t_1\rangle^{1+\alpha} |(\psi'_{t_2}, \theta_t(x)\psi_{t_1})| \leq C \tag{A-3}$$

$$\int_{tm_\theta}^\infty \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1 \, \langle t_2 - m_\theta t\rangle^{\alpha} |(\psi'_{t_2}, \theta_t(x)\psi_{t_1})| \leq C \tag{A-4}$$

where C depends on θ and m_θ (in particular it diverges when $m_\theta \to \sup|\mathrm{Supp}\theta|$).

With the help of Lemma A.2, we can establish the claims of Proposition 2.5 with $b = \theta(x/t_c)$: Since the symbol t has a different meaning in the present Appendix from that in the main text, let us denote by t_* the one in the main text, i.e. the final time. Item 4) follows from (A.2) by using the bound (A-3) and the identification $t_1 = t_* - u, t_2 = t_* - v$ and $\psi = \psi' = \phi$ (for $v, u \geq 0$) and $t_1 = t_*, t_2 = t_* - v$ and $\psi_1 = \psi_\ltimes, \psi_2 = \phi$ (for $v \geq 0 > u$), and $t_1 = t_2 = t_*$ and $\psi_1 = \psi_2 = \psi_\ltimes$ (for $0 > v, u$), and $t = t_c$ in all cases. Item 5) follows with the same identifications from (A-4). Item 3) in Proposition 2.5 is addressed in Section A.2, with the same identifications as above. The same proof, though simpler, applies also to item 2).

A.1.1 Proof of the bound (A-3) in Lemma A.2

The space $\mathfrak{h}_\alpha$ was defined to consist of functions with compact support, but this is not necessary, provided one adds some polynomial decay in $|k|$ on the RHS of (1.8) (with a sufficiently large power). In the proof that follows, we will simply put such polynomial bounds, e.g. the factors $\langle\omega_i\rangle^{-2}$ in (A-9). We have

$$(\psi'_{t_2}, \theta_t\psi_{t_1}) = \int \mathrm{e}^{\mathrm{i}(|k_2|t_2 - |k_1|t_1)} \hat{\theta}_t(k_1 - k_2)\overline{\hat{\psi}'(k_2)}\hat{\psi}(k_1)\mathrm{d}k_1\mathrm{d}k_2 \tag{A-5}$$

$$= \int_0^\infty \mathrm{d}\omega_1 \int_0^\infty \mathrm{d}\omega_2 \mathrm{e}^{-\mathrm{i}(\omega_+ t_- + \omega_- t_+)} K(\omega_1, \omega_2) \tag{A-6}$$

where $\omega_\pm := \omega_1 \pm \omega_2$, $t_\pm = (t_1 \pm t_2)/2$ and

$$K(\omega_1, \omega_2) = (\omega_1\omega_2)^{d-1} \int_{S^{d-1}} \mathrm{d}\hat{k}_1 \int_{S^{d-1}} \mathrm{d}\hat{k}_2 \overline{\hat{\psi}(\omega_1\hat{k}_1)}\hat{\psi}'(\omega_2\hat{k}_2) t^d \zeta(t^2(\omega_1\hat{k}_1 - \omega_2\hat{k}_2)^2)$$

where by rotation invariance of θ we have written it as $\hat{\theta}(k) = \zeta(k^2)$ where ζ satisfies

$$|\partial^n\zeta(x)| \leq C(n, N)\langle x\rangle^{-N} \tag{A-7}$$

for all $n, N > 0$.

Lemma A.3. *There is a $\beta > \alpha$ such that for $n = 0, 1, 2, 3,$*

$$|\partial^n_{\omega_+} K(\omega_1,\omega_2)| \le C(\omega_1^{-n} + \omega_2^{-n})\frac{(\omega_1\omega_2)^{\frac{d+\beta}{2}}}{\langle\omega_1\rangle^2\langle\omega_2\rangle^2}\left(\frac{t(1/t+\omega_+)^{1-d}}{\langle\frac{1}{2}t^2\omega_-^2\rangle^{N/2}} + \frac{t(1/t+|\omega_-|)^{1-d}}{\langle\frac{1}{2}t^2\omega_+^2\rangle^{N/2}}\right) \tag{A-8}$$

Proof. Let $\hat{k}_1\cdot\hat{k}_2 =: \cos\vartheta$ with $\vartheta \in [0,\pi]$. Then,

$$\omega_1^2 + \omega_2^2 - 2\omega_1\omega_2\cos\vartheta = \cos^2(\tfrac{\vartheta}{2})\omega_-^2 + \sin^2(\tfrac{\vartheta}{2})\omega_+^2 =: Z(\omega_+,\omega_-,\vartheta)$$

and hence

$$K(\omega_1,\omega_2) = t^d\int_0^\pi \mathrm{d}\vartheta\ (\sin\vartheta)^{d-2}\zeta(t^2 Z(\omega_+,\omega_-,\vartheta))G(\omega_1,\omega_2,\vartheta)$$

with

$$G(\omega_1,\omega_2,\vartheta) = (\omega_1\omega_2)^{d-1}\int_{S^{d-1}}\mathrm{d}\hat{k}_1\int_{S^{d-2}}\mathrm{d}\hat{p}\ \overline{\hat{\phi}(\omega_1\hat{k}_1)}\hat{\phi}(\omega_2\hat{k}_2)$$

where $\hat{k}_2 = \sin\vartheta\hat{p} + \cos\vartheta\hat{k}_1$ (and $\hat{p}\perp\hat{k}_1$). Since $\psi,\psi' \in \mathfrak{h}_\alpha$,

$$|\partial^{n_1}_{\omega_1}\partial^{n_2}_{\omega_2}G(\omega_1,\omega_2,\vartheta)| \le C\prod_{i=1}^2\omega_i^{\frac{d+\beta}{2}-n_i}\langle\omega_i\rangle^{-2} \tag{A-9}$$

uniformly in ϑ. This implies

$$|\partial^n_{\omega_+}G(\omega_1,\omega_2,\vartheta)| \le C(\omega_1^{-n}+\omega_2^{-n})\prod_{i=1}^2\omega_i^{\frac{d+\beta}{2}}\langle\omega_i\rangle^{-2}$$

From (A-7) we deduce (we abbreviate $Z = Z(\omega_+,\omega_-,\vartheta)$)

$$|\partial^n_{\omega_+}\zeta(t^2Z)| \le C\omega_+^{-n}\langle t^2 Z\rangle^{-N}.$$

Combining the two previous inequalities, we get

$$|\partial^n_{\omega_+}K(\omega_1,\omega_2)| \le C(\omega_1^{-n}+\omega_2^{-n})\frac{(\omega_1\omega_2)^{\frac{d+\beta}{2}}H(\omega_1,\omega_2)}{\langle\omega_1\rangle^2\langle\omega_2\rangle^2} \tag{A-10}$$

with

$$H(\omega_1,\omega_2) = t^d\int_0^\pi\mathrm{d}\vartheta\ (\sin\vartheta)^{d-2}\langle t^2 Z\rangle^{-N}.$$

For $\vartheta \in [0,\frac{\pi}{2}]$ we have

$$Z \ge \tfrac{1}{2}(\omega_1-\omega_2)^2 + \tfrac{1}{4}\vartheta^2(\omega_1+\omega_2)^2$$

and so

$$\langle t^2 Z\rangle^{-N} \le \langle\tfrac{1}{2}t^2\omega_-^2\rangle^{-N/2}\langle\tfrac{1}{4}t^2\vartheta^2\omega_+^2\rangle^{-N/2}$$

and for $\vartheta \in [\frac{\pi}{2},\pi]$

$$\langle t^2 Z\rangle^{-N} \le \langle\tfrac{1}{2}t^2\omega_+^2\rangle^{-N/2}\langle\tfrac{1}{4}t^2(\pi-\vartheta)^2\omega_-^2\rangle^{-N/2}$$

Since

$$\int_0^{\pi/2} \mathrm{d}\vartheta \; (\sin\vartheta)^{d-2} \langle \tfrac{1}{4} t^2 \vartheta^2 \omega_+^2 \rangle^{-N/2} \le C(1 + t\omega_+)^{1-d}$$

and similarly for the integral over $[\pi/2, \pi]$ we get

$$H(\omega_1, \omega_2) \le \frac{t(1/t + \omega_+)^{1-d}}{\langle \frac{1}{2} t^2 \omega_-^2 \rangle^{N/2}} + \frac{t(1/t + \omega_-)^{1-d}}{\langle \frac{1}{2} t^2 \omega_+^2 \rangle^{N/2}}$$

which yields the claim upon substitution in (A-10).

□

Lemma A.3 implies that the functions $\omega_+^{1-\beta} \partial^2_{\omega_+} K(\omega_1, \omega_2)$ and $\omega_+^{\beta} \partial^3_{\omega_+} K(\omega_1, \omega_2)$ are integrable and $\partial_{\omega_+} K(\omega_1, \omega_2)$ vanishes if $\omega_1 = 0$ or $\omega_2 = 0$. Hence

$$(t_2 - t_1)^2 (\psi'_{t_2}, \theta_t \psi_{t_1}) = -4 \int_0^\infty \mathrm{d}\omega_1 \int_0^\infty \mathrm{d}\omega_2 \mathrm{e}^{-\mathrm{i}(\omega_+ t_- + \omega_- t_+)} \partial^2_{\omega_+} K(\omega_1, \omega_2).$$

By Lemma A.1 with $d = 1$, we get, with $0 < \gamma < 1$,

$$|(\psi'_{t_2}, \theta_t \psi_{t_1})| \le C(t_2 - t_1)^{-2-\gamma} \int_0^\infty \mathrm{d}\omega_1 \int_0^\infty \mathrm{d}\omega_2 \; \left(|\omega_+^{-\gamma} \partial^2_{\omega_+} K(\omega_1, \omega_2)| + |\omega_+^{1-\gamma} \partial^3_{\omega_+} K(\omega_1, \omega_2)| \right)$$

provided that the RHS is finite, which we prove now. The contribution of the first term in the second parenthesis in (A-8) is dominated by

$$\int_0^\infty \mathrm{d}\omega_1 \int_0^\infty \mathrm{d}\omega_2 \frac{\omega_1^{\frac{d+\beta}{2} - 2 - \gamma} \omega_2^{\frac{d+\beta}{2}}}{\langle \omega_1 \rangle^2 \langle \omega_2 \rangle^2} \frac{t(1/t + \omega_+)^{1-d}}{\langle \frac{1}{2} t^2 \omega_-^2 \rangle^{N/2}} \quad + \quad (\omega_1 \leftrightarrow \omega_2) \tag{A-11}$$

where $(\omega_1 \leftrightarrow \omega_2)$ stands for the same term but with ω_1, ω_2 interchanged. Since this term is treated in the same way, we drop it. Then, the ω_2 integral in (A-11) gives the bound

$$\int_0^\infty \mathrm{d}\omega_2 \frac{\omega_2^{\frac{d+\beta}{2}}}{\langle \omega_2 \rangle^2} \frac{t(1/t + \omega_+)^{1-d}}{\langle \frac{1}{2} t^2 \omega_-^2 \rangle^{N/2}} \le C(\omega_1 + 1/t)^{\frac{2+\beta-d}{2}}.$$

Indeed, for large N the $\langle \frac{1}{2} t^2 \omega_-^2 \rangle^{-N/2}$ factor fixes $\omega_2 = \omega_1 + \mathcal{O}(1/t)$. Therefore, we have

$$\text{(A-11)} \le C \int_0^\infty \mathrm{d}\omega_1 \frac{\omega_1^{\beta - \gamma - 1}}{\langle \omega_1 \rangle^2} \left(\frac{\omega_1}{\omega_1 + 1/t} \right)^{\frac{d-2-\beta}{2}} \le C$$

(uniformly in t) if $\gamma < \beta$ because $d \ge 3$.

The second term between brackets in (A-8) is bounded uniformly in t for $\gamma < \beta$, so the bound (A-3) is proven.

A.1.2 Proof of the bound (A-4) in Lemma A.2

For $\psi \in \mathfrak{h}_\alpha$, we write

$$\psi_t(x) = \int_{S^{d-1}} \mathrm{d}\hat{k} \int_0^\infty \mathrm{d}\omega\, \omega^{d-1} \mathrm{e}^{\mathrm{i}\omega(t-\hat{k}\cdot x)} \hat{\psi}(\omega \hat{k}),$$

and, by Lemma A.1, for some $\beta > \alpha$,

$$|\psi_t(x)| \leq C \int_{S^{d-1}} \mathrm{d}\hat{k}\, \langle t - \hat{k}\cdot x\rangle^{-d+\frac{1-\beta}{2}}.$$

Therefore, for $\psi, \psi' \in \mathfrak{h}_\alpha$

$$|(\psi_{t_1}, \theta_t(x)\psi'_{t_2})| \leq C t^d \langle t_1 - m_\theta t\rangle^{-d+\frac{1-\beta}{2}} \langle t_2 - m_\theta t\rangle^{-d+\frac{1-\beta}{2}}$$

since $|\hat{k}\cdot x| \leq m_\theta t$ on the support of θ_t, by the definition of m_θ. Combined with (A-3), this yields (A-4).

A.2 Correlation functions with momentum cutoff

We treat the case where $b = \theta(k/\delta)$, i.e. in momentum space.

Lemma A.4. *Let $\psi, \psi' \in \mathfrak{h}_\alpha$, then there is a $\beta > \alpha$ such that for any $\gamma, \gamma' \geq 0$ with $\gamma + \gamma' \leq \beta$;*

$$|(\psi_{t_1}, \theta(k/\delta)\psi'_{t_2})| \leq C\delta^{\gamma'} \langle t_2 - t_1\rangle^{-(2+\gamma)} \tag{A-12}$$

Proof. The LHS of (A-12) is bounded by $\int_{S^{d-1}} \mathrm{d}\hat{k} \left| \int_0^\infty \mathrm{d}\omega f(\omega\hat{k}) \mathrm{e}^{\mathrm{i}\omega(t_2-t_1)} \right|$ where

$$f(k) := k^{d-1}\theta(k/\delta)\overline{\hat{\psi}(k)}\hat{\psi}'(k), \qquad \psi, \psi' \in \mathfrak{h}_\alpha. \tag{A-13}$$

We bound, for $n = 0, 1, 2, 3$,

$$|\partial_\omega^n f(\omega\hat{k})| \leq C\omega^{1+\beta-n}\nu(\omega/\delta), \tag{A-14}$$

where $\nu(\omega) = \max(\theta(\omega) + |\partial\theta(\omega)| + |\partial^2\theta(\omega)|)$ (we wrote $\theta(|k|) = \theta(k)$ because of spherical symmetry) and we used that $\nu(\omega) = 0$ for $\omega > 1$.

Since $\partial f(\omega), \partial^2 f(\omega)$ are integrable by the bounds (A-14), we get by integration by parts

$$-(t_2 - t_1)^2 \int_0^\infty \mathrm{d}\omega f(\omega\hat{k}) \mathrm{e}^{\mathrm{i}\omega(t_2-t_1)} = \int_0^\infty \mathrm{d}\omega \partial_\omega^2 f(\omega\hat{k}) \mathrm{e}^{\mathrm{i}\omega(t_2-t_1)} \tag{A-15}$$

Furthermore, we have (uniformly in $\hat{k}$)

$$\|\omega^{1-\gamma}\partial_\omega^2 f(\omega\hat{k})\|_{L^1(\mathbb{R}_+,\mathrm{d}\omega)} + \|\omega^{-\gamma}\partial_\omega^3 f(\omega\hat{k})\|_{L^1(\mathbb{R}_+,\mathrm{d}\omega)} \leq C\delta^{\gamma'} \tag{A-16}$$

We can now apply Lemma A.1 to the function $\partial_\omega^2 f(\omega\hat{k})$ and we get the required bound. $\square$

As described following Lemma A.2, the above Lemma A.4 yields items 4) and 2) of Proposition 2.5.

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